

Quantization via Linear Homotopy Types

Urs Schreiber

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Abstract

In the foundational logical framework of homotopy-type theory we discuss a natural formalization of secondary integral transforms in stable geometric homotopy theory. We observe that this yields a process of non-perturbative cohomological quantization of local pre-quantum field theory; and show that quantum anomaly cancellation amounts to realizing this as the boundary of a field theory that is given by genuine (primary) integral transforms, hence by linear polynomial functors.

Recalling that traditional linear logic has semantics in symmetric monoidal categories and serves to formalize quantum mechanics [Ye90, Pr92, AbDu05, Du06], what we consider is its refinement to *linear homotopy-type theory* with semantics in stable ∞ -categories of bundles of stable homotopy types (generalized cohomology theories) formalizing Lagrangian quantum field theory, following [Sch13a, Nuiten13] and closely related to [Ha13, HopkinsLurie14].

For the reader interested in technical problems of quantization we provide non-perturbative quantization of Poisson manifolds and of the superstring; and find insight into quantum anomaly cancellation, the holographic principle and motivic structures in quantization. For the reader inclined to the interpretation of quantum mechanics we exhibit quantum superposition and interference as existential quantification in linear homotopy-type theory. For the reader inclined to foundations we provide a refinement of the proposal in [Law91, Law94] for a formal foundation of physics [Law86, Law97], lifted from classical continuum mechanics to local Lagrangian quantum gauge field theory.

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1 Introduction and Summary

1.1 Introduction

The modern theory of fundamental physics is local quantum field theory (local QFT). The axiomatic characterization of local QFT on spacetimes/worldvolumes of arbitrary topology (see [SaSc11] for survey and review) had been completed in [Lurie09a] via a universal construction in monoidal higher category theory. Here the “higher” categorical structures are a direct reflection of the locality of the QFT. While in the “bulk” of the worldvolume this describes topological QFT (TQFT, depending on the topology of the world-volume but not on any geometric structure on it), the axiomatization also captures defects (domain walls) and boundaries (branes) and the corresponding boundary field theories may be geometric (non-topological) [Fre12, Sch13a]. In particular quantum mechanics is a boundary field theory of the non-perturbative 2d Poisson-Chern-Simons TQFT in this way [Bongers13, Nuiten13].

But modern quantum physics is more than quantum mechanics, and the local quantum field theories of interest both in nature and in theory are not random examples of the axioms of local QFT, instead they are supposed to arise via a process of “quantization” from geometric data, namely from higher pre-quantum geometry [FRS13]. Here now the “higher” geometry (hence geometric homotopy theory, higher topos theory) is the reflection of the locality of the pre-quantum field theory.

Hence the process of quantization is central to modern fundamental physics and the areas of mathematics influenced by it – but the mysteries involved in making full formal sense of it are proverbial. At least the perturbative approximation (power series expansion) to quantization of Poisson manifolds, hence of systems of phase spaces¹ has been understood in [CaFe00] as being the boundary field theory of the perturbative 2d Poisson field theory. The refinement of this from perturbative mechanics to perturbative field theory is currently being investigated in terms of “factorization algebras” by Costello et. al. On the other hand, the full (non-perturbative) quantization of symplectic manifolds (phase spaces) via Kostant-Souriau geometric quantization had been observed by Bott to be given by index theory in K-theory². A refinement of this from mechanics to field theory has been proposed and studied in [Sch13a, Nuiten13] in terms of index theory in the twisted generalized cohomology of cohesive homotopy types.

This note here describes how this cohomological (or “motivic”, see 8.2 below) non-perturbative quantization of local boundary pre-quantum field theory is naturally and usefully formalized in the logical framework of type theory – in its dependent and intensional flavor called *homotopy type theory* [UFP13]. The axiomatization is meant to provide clarity where proverbial mysteries have prevailed and seems to justify itself by its results and implications.

The process is naturally divided into two stages: first, pre-quantum geometry (involving local Lagrangians and phase spaces) is naturally axiomatized in *cohesive homotopy type theory* [ScSh12]; second, quantization (geometric quantization and path integral quantization, in fact we find a subtle mix of both) is naturally axiomatized in *linear homotopy-type theory* (see 3.2 below). In fact we find that linear homotopy-type theory provides an improved *quantum logic* that, contrary to the common perception of traditional quantum logic, indeed serves as a powerful tool for reasoning about what is just as commonly perceived as the more subtle aspects of quantum theory, such as the path integral, quantum anomalies, holography, motivic structure.

The first step has been discussed in detail in [Sch13a, ScSh12], see [Sch13b] for a quick exposition. Aspects of the second step have been worked out in examples in [Nuiten13]. There however a more fundamental type-theoretic formalization is not made explicit yet. The present note essentially points out how the quantization process in section 4 of [Nuiten13] has a natural abstract formalization in linear homotopy-type theory which provides deeper insights.

pre-quantum geometry	quantization
cohesive homotopy-type theory	linear homotopy-type theory

A brief word on the role of “homotopy”: This is often regarded as a fancy new ingredient, but the truth is

¹see ncatlab.org/nlab/show/deformation+quantization for survey and references

²see ncatlab.org/nlab/show/geometric+quantization for survey and references

that it is at the very heart of modern physics. The *gauge principle* in physics says that it is wrong to identify any two field configurations, that instead one is to ask if there is a specified “gauge” equivalence relating them. Any two gauge equivalences in turn may be related themselves by higher order gauge equivalences. This state of affairs concerning identity/equivalence is exactly what makes “homotopy type theory” be about homotopy types. That homotopy theory can be at the very heart of modern fundamental physics and still be largely under-appreciated among researchers has a simple reason: since quantum field theory is demanding, it is mostly considered only in perturbative (infinitesimal) approximation, a limit in which much of its genuine structure can be ignored. (What physicist call the “BRST complex” is the infinitesimal approximation to the geometric homotopy-type of gauge field configurations.) For instance most (if not all) mathematical physics textbooks agree that a field in physics is to be formalized as a section of some fiber bundle called the “field bundle”. But for gauge field theories, and hence for the majority of all field theories of interest, this statement makes sense only perturbatively, while non-perturbatively it is plain wrong: a gauge field is itself a fiber bundle equipped with connection, and there is in general no bundle such that its space of sections is equivalently the space of gauge fields. (This is true only perturbatively, when one fixes one background gauge field and considers only a neighbourhood of that.) Instead, the space of gauge fields is the space of sections of what is called a “2-bundle” [NSS12], hence a *stack*, a higher geometric homotopy type. Non-perturbatively one can only avoid the need for homotopy types if one breaks locality (a stack is a *local* assignment of homotopy types).³ In more refined situations this state of affairs becomes only more pronounced. For instance the full non-perturbative description of higher gauge fields, such as the “RR-field” [DFM12], without homotopy theory is simply out of the question, see 2.3.3 below. Indeed, claim 2.15 below says that gauge field theory is identified with homotopy type theory qualified by a certain dual pair of modalities.

The formalization presented here emerged in the course of attacking a class of open problems in modern quantum field theory: Non-perturbative local quantum physics has subtle consistency conditions on well-definedness of its action functionals called “quantum anomaly cancellation” conditions, see [Fre86] for a clean account of the traditional story and see 7.3 below for our general abstract formulation. By the discussion there, these anomalies are obstructions to defining action functional data globally on moduli stacks of physical fields, hence on geometric homotopy types. A famous example in 2-dimensional QFT is known as the Freed-Witten-Kapustin anomaly (see example 7.5 below); in its full form this requires a field called the RR-field to be given by cocycles in twisted differential K-theory [DFM12]. This anomaly is at the heart of all discussion of Yang-Mills gauge field theory [JaWi] from open strings on D-branes (see e.g. [Moo05]), hence of AdS/CFT duality (see. e.g. [Na07] for the traditional story and see 8.1 below for our abstract formulation) and it affects the consistency of points in the moduli space of string theory vacua, the “landscape of string vacua” [Sch106]. Despite this importance, the definition and construction of cocycles and moduli stacks of twisted differential K-theory used to be elusive. A resolution was only recently given, in terms of cohesive homotopy theory see [BNV13] and section 4.1.2 of [Sch13a]. Quantum anomalies in higher dimensional field theory are expected to be similarly expressed in terms of twisted differential generalized cohomology of higher chromatic degree [Sa11]. For more discussion of these applications see section 5 of [Sch13a].

It is for such purposes of problem solving in QFT that we develop the present formalism; but the more philosophically inclined reader (and all others should skip ahead) might appreciate that our formalization has something to offer beyond just technical problem solving. It may also be regarded as resolving some of the infamous “problems of interpretation” of quantum physics, by translating them from inadequate common language to a genuinely well adapted formal logic. For instance in the discussion below, the path integral turns out to be given essentially by the linear homotopy theoretic refinement of the existential quantifier \exists , the linear dependent sum \sum . Therefore our discussion may be regarded as providing a formal and useful sense in which the statement “There is a physical trajectory.” of classical logic first turns into “The space of all trajectories.” in homotopy type theory and then into “The linear space of quantum interfering trajectories.” in linear homotopy-type theory. This is the content of remark 4.21 below. This might be thought of as a

³Recently when the description of Yang-Mills gauge theory was undertaken, using the non-perturbative methods of “algebraic quantum field theory”, just this breaking of locality was found to be a consequence of assuming an ordinary field bundle instead of a higher homotopy type of fields, in [BDS13].

mathematical resolution of the interpretation of quantum interference via the non-classical formal meaning of “existence” in linear homotopy-type theory.

For readers with even more tolerance for metaphysics (and all others are urged to skip this) we indicate below how the axioms in type theory that we base our discussion on have a striking resemblance to core parts of the metaphysics laid out in [Heg12], once one posits, following [Law91, Law94], that the “unity of opposites” in which this is formulated is to be formalized by adjoint pairs of higher modalities [Shul12b] in type theory, see 2.3 below.

1.2 Summary (point of view of Foundations)

This document is an expanded set of notes for some talks I gave in February 2014. This section here is the script for what I actually said in a talk at the meeting “Philosophy of Mechanics: Mathematical Foundations” at Paris Diderot, to an audience of mathematical physicists interested in the mathematical and philosophical foundations of quantum physics. (Another summary aimed instead at TQFT theoreticians is below in 1.3. A summary of more technical details is below in 1.4.)

1.2.1 Categories of homotopy-types and The gauge principle

We will be considering a formal system that refines traditional quantum logic such as to allow it to speak about quantization of field theories. Since the proverb has it that quantization is a mystery, such a formalization inevitably tends to raise the question of whether it helps with “interpreting” the theory (in the physicist’s vague sense, not the logician’s precise one). That is not our actual concern, but let it here serve as a lead-in.

Ever since Galileo, mathematical formalization alone serves to clarify and demystify physics⁴). The ancients found the daily rising of the sun a mystery, some found it non-evident enough to sacrifice their own kin in the hope to propel the process. Later people were still mystified by the epicyclic intransparency of heavenly motions. That people tend to not be mystified by any of this anymore is not because we now have some deep “interpretation” of the concept of moving point masses acted on by forces-at-a-distance; instead, we just found a formal system (Newton’s equations of motion, to start with) that naturally and transparently allows to deduce these processes; and after staring at that for a while and finding all the useful facts it implies, it began to look very much self-evident.

The same ought to be true for a working formal quantum physics, a “quantum logic”. A working quantum logic should be a formalism that is more than the “QM 101 made difficult” as which traditional quantum logic must appear (see 3.1 below for review); it should instead be a formalism that empowers us to think useful thoughts that were previously hard to think and that inform us about the genuine deep aspects of quantum physics. To recall some of these:

Deep structural aspects of quantum physics.

1. the path integral;
2. quantum anomaly cancellation;
3. non-perturbative effects;
4. holography;
5. motivic structures.

This points to what is a well-kept secret in much of the literature on quantum logic and related issues: Quantum mechanics is not actually our most fundamental theory of nature. Instead, modern physics says that reality is fundamentally governed by *quantum field theory*. More specifically, modern physics is based on *local Lagrangian gauge quantum field theory*.

⁴ See also J. Butterfield’s talk [Bu14] at the same meeting.

Characteristics of fundamental quantum physics.

1. **fields** – types of configurations depending on n -dimensional spaces;
2. **gauge** – types of fields are really *moduli stacks*, geometric homotopy types;
3. **Lagrangian** – obtained via quantization from Lagrangian n -cocycle data;
4. **local** – n -dimensional theory is an n -categorical construction.

Quantum mechanics itself is just one limiting case of that. But if there is interesting formal structure in the foundations of physics at all, then it seems plausible that these are most purely exhibited by the very foundations, and not so much by some special limiting case.

The other proverb, the one referring to the effectiveness of mathematics in the physical sciences, suggests that a foundations of fundamental physics should go along well with the very foundations of mathematics and logic themselves. These have seen some considerable advances since, say, Hilbert, and we are going to take these into full account.

Logic sits inside type theory. A deep insight (attributed to Brouwer-Heyting-Kolmogorov, also to Howard) says that propositional or first order logic may naturally be regarded as a subsystem of what more fundamentally is *type theory* or *dependent type theory*, respectively (see 2.1 below for a little bit more on this). Under this *BHK correspondence* a proposition ϕ about terms x of type X is identified as the sub-type

$$\begin{array}{ccc} \sum_{x \in X} \phi(x) = \{x \in X \mid \phi(x) \text{ true}\} & & \\ \downarrow & & \in \mathbf{H}_{/X} \\ X & & \end{array}$$

of X of all those terms x which validate ϕ . Here \mathbf{H} denotes the *category* of all types – and that is the insight of *categorical logic*, that the syntax of type theories has semantics in suitable categories. Moreover, $\mathbf{H}_{/X}$ denotes the slice category over the type X , and this is what interprets X -*dependent* types.

It should be plausible that using types instead of (just) propositions as the fundamental logical substrate suits formalization of fundamental physics, for here we are clearly concerned with talking not just about propositions, but about “things”, notably when talking about some *type of fields*, which we generically write

- **Fields** $\in \mathbf{H}$ – a moduli space of fields.

And indeed these should depend on other types, such as on spacetimes $X \in \mathbf{H}$, for instance in order to form the *field bundle*

$$\begin{array}{ccc} \mathbf{Fields}_X & & \\ \bullet \quad \downarrow & \in \mathbf{H}_{/X} \text{ – a bundle of moduli of fields parameterized over } X & \\ X & & \end{array}$$

of which the actual field configurations of a field theory on X would be sections.

Dependent types and Existence. The only basic operations on (dependent) types are these: for any morphism $f : X \rightarrow Y$ in \mathbf{H} (hence a function sending terms of type X to terms of type Y) there is an adjoint triple

$$\left(\sum_f \dashv f^* \dashv \prod_f \right) : \mathbf{H}_{/X} \begin{array}{c} \xrightarrow{\Sigma_f} \\ \xleftarrow{f^*} \\ \xrightarrow{\Pi_f} \end{array} \mathbf{H}_{/Y}$$

whose operations are called, in type theory and in order of appearance, the dependent sum, the context extension and the dependent product along f . (In geometry and topos theory this is instead known as base

change, or similar.) Here sum and product are to be understood fiberwise over the fibers of f , and so if we think of a bundle E over X as a collection of types $E(x)$ for $x \in X$, then the dependent sum reads

$$(\sum_f E)(y) = \sum_{x \in f^{-1}(y)} E(x)$$

and is hence manifestly a form of fiber integration along f . This is going to play a role in the path integral below (see 4.4 for more).

We already used the \sum -notation above for indicating how propositions appear as types, and indeed the combined restriction and co-restriction of $\sum_X : \mathbf{H}_{/X} \rightarrow X$ to propositions is existential quantification:

$\exists_{x \in f^{-1}(y)} \phi(x)$. So where propositional logic has the proposition “There exists an x such that $\phi(x)$ is true.” its embedding into type theory replaces that with “The collection of all x such that $\phi(x)$ is true.” Something to keep in mind when we get to the path integral below.

Constructive \leftrightarrow Physically realizable. Closely related is the fact that type theory embodies *constructive mathematics*, where nothing is regarded as true unless its proof may be constructed in a way that yields an algorithm. For instance in type theory to prove that there exists x such that $\phi(x)$ is true is to actually construct a term $t \in \sum_{x \in X} \phi(x)$ from the deductive rules of the theory. This explains the fundamental relevance of type theory in computer science, where these proofs are the very programs – which shows that the constructive concept of existence is in some way closely related to the physical concept of existence.

Taking this constructivism fully seriously lead to a breakthrough convergence of formerly disparate concepts:

Constructive identity types \leftrightarrow Gauge principle. Constructivism demands that given a (dependent) type \mathbf{Fields}_X (of fields, for instance) and given two terms ϕ_1, ϕ_2 of that type (so: two field configurations of given type over a spacetime X) then it is misguided to ask whether these are equal or not, instead we have to construct a witness α exhibiting their equivalence, hence produce a *gauge transformation* making them gauge equivalent. It is in turn wrong to assert that two such gauge equivalences α_1 and α_2 are equal, instead we have to exhibit a gauge-of-gauge transformation between those:

$$\begin{array}{ccc} & \alpha_1 & \\ \curvearrowright & \Downarrow \simeq & \curvearrowleft \\ \phi_1 & & \phi_2 \\ \curvearrowleft & \Updownarrow \simeq & \curvearrowright \\ & \alpha_2 & \end{array} .$$

And so on. This is really the *gauge principle* in physics. At the same time, this is how constructive type theory is *automatically* a theory of homotopy types (of ∞ -groupoids).

Example [Sch13a]. For the purpose of the following exposition, a running example for \mathbf{H} to keep in mind is

$$\mathbf{H} = \text{SynthDiff}\infty\text{Grpd} := \text{Sh}_\infty(\text{FormMfd})$$

the homotopy topos of sheaves of homotopy types on formal smooth manifolds. This is a homotopy-theoretic version of a topos that interprets synthetic differential geometry [Law97].

Noteworthy geometric homotopy types that we encounter below are

$$\mathbf{B}^n U(1) \in \mathbf{H}$$

which are obtained by delooping the abelian Lie group $U(1) = \mathbb{R}/\mathbb{Z} \in \text{Grp}(\mathbf{H})$ n times. These are the moduli for instanton sectors of n -form $U(1)$ -gauge fields.

Notice a basic fact of homotopy theory, a first little hint of holography: Giving a homotopy as on the left of

$$\begin{array}{ccc}
 \prod_{\Sigma} \mathbf{Fields}_{\Sigma} & & \prod_{\Sigma} \mathbf{Fields}_{\Sigma} \\
 \swarrow & \searrow & \swarrow \downarrow \searrow \\
 * & \xrightarrow{\quad \quad} & * \\
 \searrow & \swarrow & \swarrow \downarrow \searrow \\
 \mathbf{B}^{n+1}U(1) & & \mathbf{B}^{n+1}U(1)
 \end{array}
 \quad \simeq \quad
 \begin{array}{ccc}
 \prod_{\Sigma} \mathbf{Fields}_{\Sigma} & & \prod_{\Sigma} \mathbf{Fields}_{\Sigma} \\
 \swarrow & \searrow & \swarrow \downarrow \searrow \\
 * & \xrightarrow{\quad \quad} & * \\
 \searrow & \swarrow & \swarrow \downarrow \searrow \\
 \mathbf{B}^{n+1}U(1) & & \mathbf{B}^{n+1}U(1)
 \end{array}$$

is equivalently a dashed map as shown on the right. If here we think of the tip as a type of fields over a closed manifold Σ , and if we furthermore restrict to $n = 0$ in which case $\mathbf{B}^0U(1) \simeq U(1)$, then on the right the map denoted $\exp(\frac{i}{\hbar}S)$ may be regarded as an *action functional* on these fields, as indicated. More generally Σ may have boundaries, in which case the situation is more interesting, we come to this below.

Infinitesimal identity types \leftrightarrow homotopy Lie algebroids / BRST complexes. Homotopy types of gauge equivalences are best known in physics in the approximation of perturbation theory, where they appear as homotopy Lie algebroids known as BV-BRST complexes [HeTe92]. For instance if $\mathbf{Fields} \in \mathbf{H}$ is a type of fields and G is a gauge group acting on that, then there is the homotopy quotient $\mathbf{Fields} // G$. Infinitesimally this is the *BRST-complex*.

1.2.2 Categories of being and Prequantum geometry

To make formal sense of what we mean by “infinitesimally” here, and generally by “differential” etc. we need to equip the types with *geometric quality*. In logic and type theory such “quality” is called “modality”. For propositions a modality is “a way of being true” – for instance and traditionally: “possibly being true” or “necessarily being true”. But for types a modality is simply “a way of being”.

A modality is formalized (see 2.2 below for more) as *monad* or *comonad* on the type system \mathbf{H} . Lawvere observed [Law91] that adding an *adjoint idempotent modality*

$$\begin{array}{ccc}
 \emptyset & \subset & \flat \\
 \downarrow & & \downarrow \\
 * & \subset & \sharp
 \end{array}$$

naturally has the following meaning:

- \flat is the modality of “being geometrically discrete”;
- \sharp is the modality of “being geometrically codiscrete”.

Adding one more makes it a *category of cohesion*

$$\begin{array}{ccc} & & f \\ & & \downarrow \\ \emptyset & \subset & b \\ \downarrow & & \downarrow \\ * & \subset & \sharp \end{array}$$

In homotopy-type theory we have that

- \int is the modality of “being homotopy invariant”.

In [Sch13a] we add three more to obtain what we called a *category of differential cohesion*

$$\begin{array}{ccccccc} & & & & \mathfrak{R} & & \\ & & & & \downarrow & & \\ & & \int & \subset & \oint & & \\ & & \downarrow & & \downarrow & & \\ \emptyset & \subset & b & \subset & \mathfrak{S} & & \\ \downarrow & & \downarrow & & & & \\ * & \subset & \sharp & & & & \end{array}$$

We find that

- \oint is the modality of “being formally étale”, orthogonal to “being infinitesimal”.

Claim (Theorem) a) ([Sch13a]): In cohesive homotopy-type theory there is a natural construction of differential cohomology = higher gauge theory fields.

For instance there is the type $\mathbf{BU}(1)_{\text{conn}} \in \mathbf{H}$ characterized by the fact that for X a smooth manifold, then functions

$$\nabla : X \longrightarrow \mathbf{BU}(1)_{\text{conn}}$$

are equivalently $U(1)$ -principal connections on X – for instance electromagnetic field configurations.

Improved Claim a) ([BNV13]) In cohesive homotopy-type theory *every* stable homotopy-type represents a generalized differential cohomology theory (such as differential K-theory, differential elliptic cohomology, etc.) hence a higher gauge theory.

Claim (Theorem) b) ([FRS13], [Sch13a]): Cohesive homotopy-type theory naturally encodes local pre-quantum geometry, hence Lagrangian cocycle data on higher moduli stacks of fields.

We continue to provide some examples of this.

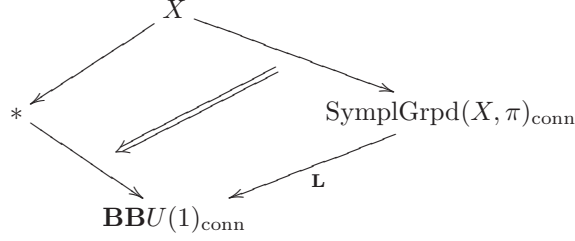
running Example a) (7.3) – The particle at the boundary of 2d Poisson-Chern-Simons theory. Consider a Poisson manifold (X, π) , the phase spaces of a foliation by mechanical systems. It is encoded in its Poisson Lie algebroid \mathfrak{P} , which is equipped with a 2-plectic cocycle π

$$\mathfrak{P} \xrightarrow{\pi} \mathbf{B}^2\mathbb{R}$$

Theorem. [FSS12][Bongers13] Higher Lie integration $\exp(-)$ of π yields the Lagrangian \mathbf{L}_{2dCS} for non-perturbative 2-d Poisson-Chern-Simons theory

$$\mathbf{L}_{2dCS} = \exp(\pi) : \text{SympGrpd}(\mathfrak{P}, \pi)_{\text{conn}} \longrightarrow \mathbf{B}^2U(1)_{\text{conn}}$$

whose moduli stack of fields is differential cohomology refinement of the “symplectic groupoid”. Moreover, the original Poisson manifold is a boundary condition of this 2-d theory exhibited by a correspondence diagram in the slice $\mathbf{H}_{\mathbf{B}(BU(1)_{\text{conn}})}$



which describes the points of X as “trajectories” along which the fields of the 2d theory may approach the boundary.

running Example b) (7.4) – The string at the boundary of 3d Chern-Simons theory.

Theorem [FSS12], exposition in [FSS13]. The Lie integration of the canonical Lie algebra 3-cocycle on the orthogonal Lie algebra

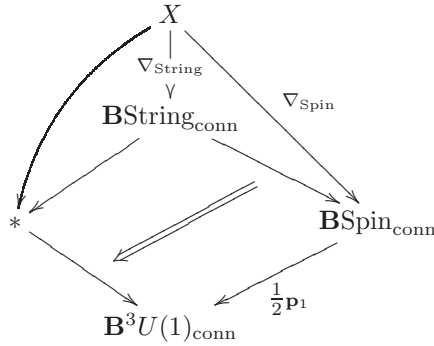
$$\mathfrak{so} \xrightarrow{\langle -, [-, -] \rangle} \mathbf{B}^2\mathbb{R}$$

is a differential cohomology refinement of the first fractional Pontryagin class $\frac{1}{2}p_1$

$$\mathbf{L}_{3dCS} = \frac{1}{2}\mathbf{p} : \mathbf{B}\text{Spin}_{\text{conn}} \longrightarrow \mathbf{B}^3U(1)_{\text{conn}}$$

and this is the local Lagrangian for 3d Spin Chern-Simons theory.

Moreover, the *universal* boundary condition for this is the delooped String 2-group $\mathbf{B}\text{String}$ (see section 5.1 of [Sch13a]), and hence a manifold X via a Spin-structure ∇_{Spin} yields a boundary for the 3d Chern-Simons theory here precisely if it lifts to a String-structure ∇_{String}



Below we see the holographic quantization of these two examples. In general, a space of field trajectories $\mathbf{Fields}_{\text{traj}}$ equipped with action functional data for an n -dimensional field theory is a correspondence in

$\mathbf{B}/\mathbf{B}^n U(1)$ of the form

$$\begin{array}{ccccc}
 & & \mathbf{Fields}_{\text{traj}} & & \\
 & \swarrow^{(-)|_{\text{out}}} & & \searrow^{(-)|_{\text{in}}} & \\
 \mathbf{Fields}_{\text{out}} & & & & \mathbf{Fields}_{\text{in}} \\
 & \swarrow^{\mathbf{L}_{\text{out}}} & \exp(\frac{i}{\hbar} S) & \searrow^{\mathbf{L}_{\text{in}}} & \\
 & & \mathbf{B}^n U(1) & &
 \end{array}$$

Now to quantize all this higher prequantum geometry.

1.2.3 Categories of linear homotopy-types and Quantization

There is a simple idea: quantization is *linearization* of the above pre-quantum geometry, analogous to how motivic geometry is a linearization of algebraic geometry. To formalize this, think of “linear” as being “affine with basepoint”.

quantum \leftrightarrow **linear** = **base-point** + **affine**. The “modality of being pointed” is the *maybe monad*

$$*/ : X \mapsto X \sqcup *$$

Indeed, the $*/$ -modal types are equivalently the pointed types canonically equipped with the smash tensor product as “linear conjunction” (3.2 below). Notice that this is a non-cartesian tensor product, the hallmark of quantum theory in category theory (see 3.1 for more on this).

Moreover, according to deformation theory a pointed space is affine if it is infinitesimally extended, which by the above means it is orthogonal to \oint -modal types.

Definition: For $X \in \mathbf{H}$ then $\text{Mod}(X)$ are the $*/$ -modal types in $\mathbf{H}/_X$ which are left orthogonal to \oint -modal types.

Proposition (3.2): this forms a *linear* homotopy-type theory

$$\begin{array}{ccc}
 & \xrightarrow{\Sigma_f} & \\
 \text{Mod}(X) & \xleftarrow{f^*} \text{Mod}(Y) & \\
 & \xrightarrow{\Pi_f} &
 \end{array}$$

Under this extension the meaning of the existential quantifier \sum changes drastically (for the following assume for simplicity of notation that $\text{Line}(*)$ has a reflection as $\text{Line} \in \mathbf{H}$ that classifies invertible linear types):

Definition (the path integral, 4.4): First turn an action functional on trajectories as above into an *integral kernel* by associating linear coefficients

$$\begin{array}{ccccc}
 & & \mathbf{Fields}_{\text{traj}} & & \\
 & \swarrow^{(-)|_{\text{out}}} & & \searrow^{(-)|_{\text{in}}} & \\
 \mathbf{Fields}_{\text{out}} & & & & \mathbf{Fields}_{\text{in}} \\
 & \swarrow^{\mathbf{L}_{\text{out}}} & \exp(\frac{i}{\hbar} S) & \searrow^{\mathbf{L}_{\text{in}}} & \\
 & & \mathbf{B}^n U(1) & & \\
 & & \downarrow & & \\
 & & \text{Line}(*) & &
 \end{array}$$

The path integral is then effectively the linear sum over this: $\sum_{\mathbf{Fields}_{\text{traj}}} \exp(\frac{i}{\hbar} S)$.

The full expression involves pre-composing this with the $(\prod_{\text{in}} \dashv \text{in}^*)$ -unit followed by a “twisted ambidexterity” measure $d\mu$, and postcomposing with the $(\sum_{\text{out}} \dashv \text{out}^*)$ -counit:

$$\mathbb{D} \int_{\mathbf{Fields}_{\text{traj}}} \exp(\frac{i}{\hbar} S) d\mu =$$

$$\sum_{\mathbf{Fields}_{\text{out}}} \mathbf{L}_{\text{out}} \xleftarrow{\sum_{\mathbf{Fields}_{\text{out}}} \epsilon_{\mathbf{L}_{\text{out}}}} \sum_{\mathbf{Fields}_{\text{out}}} \text{out}! \text{out}^* \mathbf{L}_{\text{out}} \xleftarrow{\simeq} \sum_{\mathbf{Fields}_{\text{traj}}} \text{out}^* \mathbf{L}_{\text{out}} \xleftarrow{\sum_{\mathbf{Fields}_{\text{traj}}} \exp(\frac{i}{\hbar} S)} \sum_{\mathbf{Fields}_{\text{traj}}} \text{in}^* \mathbf{L}_{\text{in}} \xleftarrow{\simeq} \sum_{\mathbf{Fields}_{\text{in}}} \text{in}! \text{in}^* \mathbf{L}_{\text{in}} \xleftarrow{\sum_{\mathbf{Fields}_{\text{in}}} [\text{in}]}} \sum_{\mathbf{Fields}_{\text{in}}} \mathbf{L}_{\text{in}} \otimes \tau$$

Notice how here what used to be existential quantification becomes the path integral with its superposition and quantum interference:

The quantum incarnation of existential quantification.

1. in logic: That there exists a path.
2. in type theory: The collection of all paths.
3. in homotopy-type theory: The collection of all paths with gauge equivalences between them.
4. in linear type theory: The linear addition (superposition) of amplitudes of all paths.
5. in linear homotopy-type theory: The linear addition (superposition) of amplitudes of all paths with gauge equivalences taken into account.

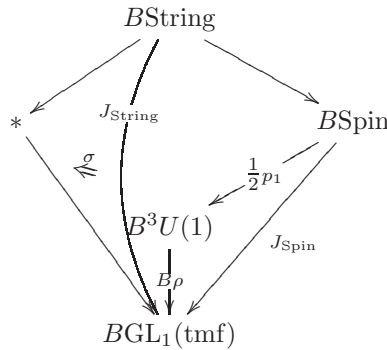
Theorem 7.3: This is quantum anomaly free if there is cobounding theory.

running example a) – The particle at the boundary of 2d Poisson-Chern-Simons theory:

Proposition [Nuiten13], 7.3: The above path integral yields the geometric quantization of symplectic manifolds in its K-theoretic incarnation (following Bott). It generalizes it to a geometric quantization of Poisson manifolds and reproduces there for instance the universal orbit method of [FHT05].

running example b) – The string at the boundary of 3d Chern-Simons theory

Proposition 7.4 Quantization of the boundary of Spin Chern-Simons with coefficients in the universal elliptic cohomology ring tmf



yields the non-perturbative refinement of the Witten genus [Wi87], the partition function of the string, to the String-orientation of tmf [AHR10]

$$\text{tmf} = \sum_* 1_* \longleftarrow \sum_{B\text{String}} J_{\text{String}} = M\text{String} \wedge \text{tmf} .$$

(This follows using section 8 of [ABG11].)

1.2.4 Outlook

We are seeing here the pattern of the holographic principle [Maldacena97, Wi98]. The next example of interest of this form is induced from a local Lagrangian for 7-dimensional Chern-Simons theory [FSS12].

field theory	spaces of states	propagator	particle/2dCS	string/3dCS	6dSCFT/7dCS
TQFT_{d+1}	$\text{Mod}_2 \in \text{Cat}_2$	integral transform	TQFT_3	TQFT_4	TQFT_8
TQFT_d^r	$\text{Mod}(\ast) \in \text{Mod}_2$	secondary integral transform	CS_2	CS_3	CS_7
QFT_{d-1}	$\sum_X A_X \in \text{Mod}(\ast)$	equivariance under Hamiltonian group action	QM_1	WZW_2	WZW_6

See also [Fre12]. Notice that according to [Wi07b] the Kaluza-Klein compactification of WZW_6 on a torus is 4-dimensional (super-)Yang-Mills theory. This way we find at least a sketch of a plausible path here for how to approach the quantization of Yang-Mills theory [JaWi] in linear homotopy-type theory. A central open question from this perspective is which brave new ring would serve as the right coefficient for quantization of the 7d Chern-Simons theory.

1.3 Summary (point of view of TQFT)

This document is an expanded set of notes for some talks I gave in February 2014. What follows here is the script for what I actually said in a talk at the “Modern Trends in TQFT” meeting at ESI in Vienna, to an audience of TQFT theoreticians. (Another summary aimed instead at mathematical physicists interested in foundations is above in 1.2. A summary of more technical details is below in 1.4.)

1.3.1 The need for Lagrangian TQFT and Quantization

Since the (very detailed outline of) the proof of the cobordism hypothesis [Lurie09a], there is a full mathematical classification of *local* quantum field theory in the form of monoidal (∞, n) -functors

$$Z : \mathcal{SBord}_n^{\sqcup} \longrightarrow \text{Mod}_n^{\otimes}.$$

(Here we denote by \mathcal{SBord}_n the symmetric monoidal (∞, n) -category of cobordisms (spacetimes, worldvolumes) equipped with some given structure \mathcal{S} – which we take to include lifts of structure groups but also singularity data – while Mod_n is some symmetric monoidal (∞, n) -category of n -categorical modules. Of course this is *topological* field theory, but one point of relevance in the following is that once singularity data is allowed, such Z contains boundary and defect data which corresponds to non-topological/geometric field theory in the sense of physics.)

While this result is a celebrated and fundamental result, it is noteworthy that the field theories appearing in nature and in physical theory are not random examples of this classification. Instead they are instances of *Lagrangian quantum field theory*: they arise from a process of “quantization” from “Lagrangian” geometric n -cocycle data.

Local Lagrangian field theory.

local pre-quantum field theory	quantization \longrightarrow	local quantum field theory
Lagrangian, action functional		propagator, correlator, S-matrix
geometric n -cocycle: $\mathbf{Fields} \xrightarrow{\mathbf{L}} \mathbf{B}^n U(1)_{\text{conn}}$		monoidal n -functor: $\mathcal{SBord}_n^{\sqcup} \xrightarrow{Z} \text{Mod}_n$

Hence the next **open question** is:

Which higher categorical refinement of the process of quantization produces such functors Z from Lagrangian

data \mathbf{L} so that we may eventually write

$$“Z \simeq \int_{\phi \in \mathbf{Fields}_{\text{traj}}} \exp(\frac{i}{\hbar} S(\phi)) d\mu”$$

with $\exp(\frac{i}{\hbar} S)$ the action functional obtained by transgression from the Lagrangian \mathbf{L} ?

Relevance in physics. To see that this is an issue relevant for mathematical/theoretical physics, the most famous example is 3-dimensional Chern-Simons gauge field theory [Wi89] for a compact Lie group G . The theory is defined to be the quantization of the Chern-Simons Lagrangian \mathbf{L}_{CS} (whose action functional sends fields given by G -principal connections to the volume holonomy of the secondary differential characteristic of their second Chern class [FSS13]). It has always been conjectured that the result of this is equivalently the functor Z_{MTC} on 3-bordisms which is constructed algebraically (via Reshetikhin-Turaev, Turaev-Viro, skein relations, etc., see for instance [Ro05] for a review) from the modular tensor category of G . This conjectured equivalence is crucial for various results that build on it, for instance the construction and classification [FRS09] of full (defined on all cobordisms) rational 2d conformal field theory CFT_{rat} as the holographic boundary theory of Chern-Simons .

$$\begin{array}{ccc} \mathbf{L}_{\text{CS}} & \xrightarrow{? \text{ quantization } ?} & Z_{\text{MTC}} \\ & & \downarrow \text{holography} \\ & & \text{CFT}_{\text{rat}} \end{array}$$

(To put this in perspective: this is the full classification of a (tiny) fraction of the the moduli space of 2d CFTs, the “landscape” of string theory [Sch06].) However, an actual proof of $\mathbf{L}_{\text{CS}} \mapsto Z_{\text{MTC}}$ had been elusive. One is claimed only rather recently⁵ in [AnUe11].

For non-compact gauge groups, the situation is even more interesting and even more subtle. In [Wi88] it was famously argued that 3d quantum gravity with suitable cosmological constant is equivalent to 3d quantum Chern-Simons theory with suitable non-compact gauge group. But later in [Wi07a] are listed arguments why this cannot quite be after all, and the issue remains open. On the answer to this it depends whether a whole sequence of mathematical results informs us about 3d quantum gravity or not, for instance Liouville field theory [CHvD95] and quantum Teichmüller theory [Te04].

$$\begin{array}{ccc} \mathbf{L}_{\text{3dGrav}} & \xrightarrow{?} & \mathbf{L}_{\text{CS}} \\ & & \downarrow \\ & & \text{CFT}_{\text{Liouv.}} \end{array}$$

And of course for the case of actual physical interest, quantum gravity in dimensions ≥ 4 the problem is, infamously, more interesting and wide open.

Here we won’t solve this problem, clearly, but we do want to start to systematically investigate it with suitable mathematical formalism.

Relation to existing literature. Proposals along the lines to follow in available literature include:

- There is a long tradition, going back to [We71] in the context of mechanics and then particularly picked up in [Fre92] in the context of field theory, to propose that the correct domain for quantization are correspondences of phases spaces (or configuration spaces) of fields, and that (geometric/path integral) quantization is a kind of linearization of such correspondences of spaces given by pull-push, much as in the theory of motives. (More on this relation below in 8.2).

⁵See also J. Andersen’s talk at the *String Geometry Network* meeting.

- For the comparatively simple case of higher gauge theories of Dijkgraaf-Witten type (with finite gauge group) a method for such pull-push quantization is sketched in sections 3 and 8 of [FHLT09]. Partial details on how to flesh out that proposal have now been made available in [HopkinsLurie14].
- Non-perturbative quantization of non-finite and non-topological theories as boundary field theories of local topological field theories has been studied in [Nuiten13]. The two main examples there we will re-consider below: the quantum particle at the boundary of the 2d Poisson-Chern-Simons theory and the quantum superstring at the boundary of the 3d Spin-Chern-Simons theory.

These proposals focus on different sub-aspects of the general issue and should be subsumed in a comprehensive theory. Here we mean to further push in this direction. We give a formalization of the path integral that subsumes the notion of integration in [HopkinsLurie14] as a special case. It generalizes it in particular by allowing certain “twists” to appear on boundary data (twists that would be anomalous in the bulk) where we find them to give the boundary field theory of a TQFT its non-topological “geometric” character as in [Nuiten13]. Finally we show that the condition that the TQFT in the bulk is quantum anomaly free means equivalently that it is itself the boundary field theory of a yet higher dimensional theory, making the non-topological field theory not just a boundary field theory but *corner field theory*. This is a pattern that has been expected in [Sa11], see also [Fre12]. For Lagrangian pre-quantum geometry we had discussed this in sections 3.9.14 and 5.7 of [Sch13a], based on [FiVa13].

What we do. Here we consider quantization for local (extended) TQFT in dimensions $(d-1, d, d+1)$, localized to the point in a way that it produces some of the corner and boundary data for a monoidal 2-functor of the form⁶

$$\begin{array}{ccccc} \mathcal{SBord}_2^{\sqcup} & \xrightarrow[\text{loc. Lagrangian}]{\exp(\frac{i}{\hbar}S)} & \text{Corr}_2(\mathbf{H})^{\otimes} & \xrightarrow[\text{quantization}]{f(-)d\mu} & \text{Mod}_2^{\otimes} , \\ & \searrow \text{local Lagrangian TQFT} & & \nearrow & \\ & Z := \int_{\phi \in \mathbf{Fields}} \exp(\frac{i}{\hbar}S(\phi))d\mu & & & \end{array}$$

where $\mathcal{S} = \{\text{corner}\}$ is the singularity datum of a *corner* (a codim- $(n-1)$ boundary ending itself on a codim- $(n-2)$ boundary) and find an interpretation as:

1. in the middle dimension this is a TQFT_d^{τ} such as 2d Poisson-Chern-Simons or 3d Spin-Chern-Simons;
2. whose boundary field theory is a non-topological QFT_{d-1} such as the quantum particle and the quantum string;
3. whose cobounding theory TQFT_{d+1} exhibits its quantum anomaly cancellation.

What we do not consider. We discuss the data needed for such a functor essentially only on the generating corner 2-cell without discussing all the conditions needed to lift this to a complete monoidal 2-functor or rather, in the end, to a monoidal n -functor. In a somewhat simpler version, the problem of extending the bulk field theory to a full $(\infty, 2)$ -functor is the topic remark 4.2.5 of [HopkinsLurie14]. Using this one may begin to see what the necessary conditions are to lift the data that we consider here to a full monoidal n -functor, but we will not look into this here.

1.3.2 Local prequantum field theory with Boundaries and Corners

The idea of local pre-quantum field theory⁷ is that where a genuine quantum field theory assigns a space of quantum states, its pre-quantum version assigns a moduli stack **Fields** of field configurations (\sim a phase space, but see below) equipped with a Lagrangian and a prequantum line bundle. And where a genuine

⁶An detailed discussion of $\text{Corr}_2(-)$ is for instance in section 10 of [DyKa12].

⁷This here is a lightning review of section 3.9.14 in [Sch13a] (inspired from [FiVa13]).

quantum field theory assigns a propagator or S-matrix, the pre-quantum field theory assigns a space of field trajectories and a transformation between the incoming/outgoing prequantum bundles.

Let \mathbf{H} be a differentially cohesive ∞ -topos [Sch13a] such as ∞ -stacks over the site of formal manifolds $\mathbf{H} = \mathrm{Sh}_\infty(\mathrm{FormMfds})$ as discussed in section 4.5 of [Sch13a]. It being differentially cohesive means that we have moduli for differential cohomology⁸ in \mathbf{H} in particular that we know how to find moduli n -stacks of $(n-1)$ -bundle gerbes with connection, which we write

$$\mathbf{B}^n U(1)_{\mathrm{conn}} \in \mathbf{H}.$$

Example. For (X_1, ω_1) and (X_2, ω_2) two symplectic manifolds (phase spaces) and $L \xrightarrow{i} (X_1 \times X_2, p_1^* \omega_1 - p_2^* \omega_2)$ a Lagrangian subspace of their product (with one of the symplectic structures taken with opposite sign), hence a *Lagrangian correspondence* [We71], defines a correspondence in $\mathbf{H}/\Omega_{\mathrm{cl}}^2$

$$\begin{array}{ccc} & L & \\ i_2 \swarrow & & \searrow i_1 \\ X_1 & & X_2 \\ \omega_1 \searrow & & \swarrow \omega_2 \\ & \Omega_{\mathrm{cl}}^2 & \end{array}.$$

A *pre-quantization* of this is a lift through the universal curvature map $F_{(-)} : \mathbf{B}U(1)_{\mathrm{conn}} \rightarrow \Omega_{\mathrm{cl}}^2$ to a correspondence of the form [Sch13b].

$$\begin{array}{ccc} & L & \\ \swarrow & & \searrow \\ X_1 & & X_2 \\ \swarrow \mathbf{L}_1 & \exp(\frac{i}{\hbar} S) & \searrow \mathbf{L}_2 \\ & \mathbf{B}U(1)_{\mathrm{conn}} & \end{array}.$$

Proposition. [FRS13] Given a prequantum line bundle $\mathbf{L} : \mathbb{R}^{2n} \rightarrow \mathbf{B}U(1)_{\mathrm{conn}}$ prequantizing the canonical phase space, then morphisms in \mathbf{H} of the form

$$\mathbf{B}\mathbb{R} \longrightarrow \mathbf{BAut}_{\mathbf{B}U(1)_{\mathrm{conn}}}(\mathbf{L})$$

which are “concrete” (something expressible using the fourth adjoint in the definition of cohesion) are equivalent to choices of Hamiltonians $H \in C^\infty(\mathbb{R}^n)$ and send $t \in \mathbb{R}$ to the prequantized Lagrangian correspondence as above with $S = \int_{0^t} L dt$ the respective Hamilton-Jacobi action, i.e. the integral over the Legendre transform L (the Lagrangian) of H .

So this expresses the core ingredients of classical mechanics. Passing here from $\mathbf{B}U(1)_{\mathrm{conn}}$ to $\mathbf{B}^n U(1)_{\mathrm{conn}}$ produces the Hamilton-De Donder-Weyl formulation of classical local field theory (see section 1.2.11 of [Sch13a] or [Sch13b]).

Example. The gauge coupling action functional S_{int} for an electromagnetically charged particle propagating on a (spacetime) manifold X with electromagnetic field given by a $U(1)$ -principal connection $\nabla : X \rightarrow$

⁸ See also U. Bunke’s talk at the “Modern Trends in TQFT” meeting.

$\mathbf{BU}(1)_{\text{conn}}$ on a $U(1)$ -principal bundle $\mathbf{L} : X \rightarrow \mathbf{BU}(1)$ is a correspondence in $\mathbf{H}/\mathbf{BU}(1)$ with tip the smooth path space of X :

$$\begin{array}{ccc}
 & X^{[0,1]} & \\
 (-)|_0 \swarrow & & \searrow (-)|_1 \\
 X & & X \\
 \mathbf{L} \searrow & \exp(\frac{i}{\hbar} S_{\text{int}}) \swarrow \! \! \! \swarrow & \searrow \mathbf{L} \\
 & \mathbf{BU}(1) &
 \end{array}$$

(the components of the homotopy filling this diagram is over each path an assignment which sends a trivialization of \mathbf{L} over the endpoints of these paths to a value in $U(1)$: the *parallel transport* of along with path with respect to ∇).

Definition. For \mathbf{H} an ∞ -topos, write $\text{Corr}_n(\mathbf{H})$ for the (∞, n) -category of n -fold correspondences in \mathbf{H} .

Fact. Every object in $\text{Corr}_n(\mathbf{H})$ is fully dualizable, and in fact is fully self-dual.

Hence for an n -dimensional prequantum field theory the action functional looks like

$$\begin{array}{ccc}
 & \mathbf{Fields}_{\text{traj}} & \\
 (-)|_{\text{out}} \swarrow & & \searrow (-)|_{\text{in}} \\
 \mathbf{Fields}_{\text{out}} & & \mathbf{Fields}_{\text{in}} \\
 \mathbf{L}_{\text{out}} \searrow & \exp(\frac{i}{\hbar} S) \swarrow \! \! \! \swarrow & \searrow \mathbf{L}_{\text{in}} \\
 & \mathbf{B}^n U(1) &
 \end{array}$$

and we may axiomatize such a pre-quantum bulk field theory as

$$\exp(\frac{i}{\hbar} S) : \mathcal{SBord}_n^{\sqcup} \longrightarrow \text{Corr}_n(\mathbf{H}/\mathbf{B}^n U(1))^{\otimes}.$$

Then the idea would be that to formalize the path integral is to produce a monoidal n -functor

$$\int (-) d\mu : \text{Corr}(\mathbf{H}/\mathbf{B}^n U(1))^{\otimes} \longrightarrow \text{Mod}_n^{\otimes}$$

then the composite $\int \exp(\frac{i}{\hbar} S) d\mu$ would be the quantized field theory.

This is essentially what we will consider, only that it turns out to be natural to first realize the action functional itself as a boundary effect of an auxiliary $(d+1)$ -dimensional theory. To that end notice that on the last few pages of [Lurie09a] there is a vast generalization of the cobordism theorem to the case of cobordisms with singularities:

Cobordism hypothesis with singularities (theorem 4.3.11 in [Lurie09a], rough paraphrase): *If \mathcal{S} is a collection of “catastrophe diagrams” characterizing types of k -dimensional singularities for $0 \leq k \leq n$, then the (∞, n) -category $\mathcal{SBod}_n^{\sqcup}$ of cobordisms with singularities of these types is the free symmetric monoidal (∞, n) -category with duals generated from the elements of \mathcal{S} regarded as k -morphisms.*

Example. $\mathcal{S} = \{(\emptyset \rightarrow *)\}$ encodes a single bulk theory $*$ with a single type of codimension $(n-1)$ -boundary. A monoidal n -functor $\mathbf{Fields} : \mathcal{SBod}_n^{\sqcup} \rightarrow \text{Corr}_n(\mathbf{H})^{\otimes}$ in this case is equivalently a choice $\mathbf{Fields}(*) \in \mathbf{H}$ of bulk fields together with a morphism from $*$ to $\mathbf{Fields}(*)$ in $\text{Corr}_n(\mathbf{H})$, hence a correspondence in \mathbf{H}

$$* \longleftarrow \mathbf{Fields}^{\partial} \longrightarrow \mathbf{Fields}(*) ,$$

hence a choice of moduli $\mathbf{Fields}^{\partial}$ of boundary fields and a choice of map from these into the bulk fields.

Example. The singularity datum that we consider in the following is an elementary *corner* (formalizing corner field theories as envisioned in [Sa11])

$$\begin{array}{ccc}
 \text{Pic} & \xlongequal{\quad} & \text{Pic} \\
 \uparrow & \swarrow \xi & \uparrow \mathbf{L} \\
 * & \xleftarrow{\text{Fields}^\partial} \text{Fields} & \\
 \parallel & & \downarrow \\
 * & \xlongequal{\quad} & *
 \end{array} ,$$

This means that the d -dimensional bulk **Fields** are realized themselves as the boundary fields of a $(d+1)$ -dimensional theory with fields $\text{Pic} \in \mathbf{H}$, so that the boundary Fields^∂ of the d -dimensional theory are now corner fields of the $(d+1)$ -dimensional theory.

A diagram showing trajectories of the d -dimensional bulk theory “approaching and then hitting the boundary” is hence of the following form

$$\begin{array}{ccccccc}
 \text{Pic} & \xlongequal{\quad} & \text{Pic} & \xlongequal{\quad} & \text{Pic} & \cdots & \\
 \uparrow & \swarrow \xi & \uparrow \mathbf{L}_{\text{out}} & \swarrow \exp(\frac{i}{\hbar} S) & \uparrow \mathbf{L}_{\text{in}} & & \\
 \widehat{\text{Pic}} & * & \xleftarrow{\text{Field}^\partial} \text{Fields}_{\text{out}} & \xleftarrow{(-)|_{\text{out}}} \text{Fields}_{\text{traj}} & \xrightarrow{(-)|_{\text{in}}} \text{Fields}_{\text{in}} & \cdots & \\
 \parallel & & \downarrow & & \downarrow & & \\
 * & \xlongequal{\quad} & * & \xlongequal{\quad} & * & \cdots &
 \end{array}$$

(We will see that composition to the right here imposes strong “anomaly cancellation” constraints on the quantization to make the composites functorial. But at the boundary itself, where no further composition is possible (to the left) these conditions are relaxed and the available choices are part of what makes the boundary theory itself “geometric” (non-topological).)

1.3.3 Linear homotopy-types and Spaces of states

Traditional geometric quantization produces vector spaces of quantum states as sections of bundles of vector spaces (“prequantum bundles”) over configuration spaces/phase spaces of fields.⁹

$$\left(\begin{array}{c} \text{prequantum line bundle} \\ \mathbf{L} \\ \text{on phase space } X \end{array} \right) \mapsto \left(\begin{array}{c} \text{vector space of states} \\ \Gamma(\mathbf{L}) \\ \text{over the point} \end{array} \right) .$$

Example. If X here is just a finite set, then $\mathbf{L} \in \text{Vect}(X)$ is just an X -parameterized collection of vector spaces, and then $\Gamma_X(\mathbf{L}) \simeq \lim_{x \in X} \mathbf{L}_x$, which is left Kan extension along the terminal map $X \rightarrow *$

$$\text{Vect}(X) \begin{array}{c} \xrightarrow{\lim} \\ \xleftarrow{\text{const}} \\ \xrightarrow{\lim} \end{array} \text{Vect}(*) = \text{Vect} .$$

Plain vector spaces over the complex or real numbers like this will be too inflexible for a decent formalization of the path integral. But more general “linear homotopy-types” will do. We axiomatize the minimum structure that we need:

⁹Of course geometric quantization produces *polarized* sections. We take *all* sections for the cobounding TQFT and find – Example a) below – that this automatically makes polarized sections appear in the boundary theory.

Definition A (2-monoidal, Beck-Chevalley) model for *linear homotopy-type theory* is a Cartesian fibration of closed symmetric monoidal $(\infty, 1)$ -categories

$$\begin{array}{c} \mathrm{Mod}(-) \\ \downarrow \\ \mathbf{H} \end{array}$$

over a Cartesian monoidal ∞ -category \mathbf{H} , that satisfies good base change in that for every map $f : X \rightarrow Y$ in \mathbf{H}

1. pullback f^* is a strong monoidal ∞ -functor;
2. which has a left and a right adjoint $(f_! \dashv f^* \dashv f_*) : \mathrm{Mod}(X) \begin{array}{c} \xrightarrow{f_!} \\ \xleftarrow{f^*} \\ \xrightarrow{f_*} \end{array} \mathrm{Mod}(Y) ;$
3. and satisfies the “projection formula” $f_!((f^* A) \otimes B) \xrightarrow{\sim} A \otimes f_! B$
4. such that
 - the Beck-Chevalley condition holds for the left adjoints, meaning that

$$X(\overset{f}{\leftarrow} Z \overset{g}{\rightarrow}) Y \mapsto g_! f^* : \mathrm{Mod}(X) \rightarrow \mathrm{Mod}(Y)$$

constitutes an $(\infty, 1)$ -functor¹⁰

$$\mathrm{Corr}_1(\mathbf{H}) \rightarrow (\mathrm{Mod}(*))\mathrm{Mod},$$

- there are natural equivalences $\mathrm{Mod}(X \times_Z Y) \simeq \mathrm{Mod}(X) \underset{\mathrm{Mod}(Z)}{\otimes} \mathrm{Mod}(Y)$.

This is one incarnation of the “yoga of six operations”, the one called a “Wirthmüller context” in [May05]. In the context of (linear) homotopy type theory [UFP13] the left adjoint $f_!$ is also written \sum_f , expressing the fact that this operation “sums up” data over the (homotopy-)fibers of f . This is a very suggestive notation for the path integral that we are after, and we will interchangeably use it below.

Proposition. For E an E_∞ -ring, and $\mathbf{H} = \infty\mathrm{Grpd}$, then the assignment

$$X \mapsto \mathrm{Mod}(X) := E\mathrm{Mod}(X) := \mathrm{Func}(X, E\mathrm{Mod})$$

which sends an ∞ -groupoid (fundamental ∞ -groupoid of a topological space) to the ∞ -category of E -module spectrum bundles (E -linear “local systems”) over X is a (2-monoidal, Beck-Chevalley) model for linear homotopy-type theory.

Definition. Given $\mathrm{Mod}(-)$, the $(\infty, 2)$ -category Mod_2 of 2-modules is $\mathrm{Mod}_2 := (\mathrm{Mod}(*), \otimes)\mathrm{Mod}$.

Now given $\mathbf{L} \in \mathrm{Mod}(\mathbf{Fields})$ regarded as a prequantum bundle, then the linear homotopy-type of (dual) prequantum states is

$$\sum_{\mathbf{Fields}} \mathbf{L} \in \mathrm{Mod}(*).$$

We want that a correspondence between field moduli induces a linear map between these spaces given by integrating over spaces of trajectories. For this path integral we first need a notion of measure to integrate against.

¹⁰The BC condition says that we get an ∞ -functor to ∞ -categories, but with the projection formula it follows that this indeed takes values in $\mathrm{Mod}(*)$ -linear ∞ -categories.

1.3.4 Fundamental classes, measures and Path integral quantization via Pull-push

We now formulate in the general context of a linear homotopy-type theory as above what it means to produce (“secondary”) integral transformations by “secondary” pull-push of linear homotopy types and discuss how this subsumes as a special cases pull-push in generalized twisted cohomology.

The operation is “secondary” as the pull-push is implemented by the units and counits of the $(f_! \dashv f^* \dashv f_*)$ -operations, instead of by these operations themselves as in familiar categorified integral transforms. Below we discuss that the “secondary” operation discussed here is precisely a boundary condition for the “primary” operation and as such in direct analogy to the traditional terminology by which the Chern-Simons functional is a “secondary characteristic” where the topological Yang-Mills functional is “primary”.

field theory	spaces of states	propagator
TQFT_{d+1}	$\text{Mod}_2 \in \text{Cat}_2$	integral transform
TQFT_d^τ	$\text{Mod}(\ast) \in \text{Mod}_2$	secondary integral transform, path integral
QFT_{d-1}	$\sum_X A_X \in \text{Mod}(\ast)$	equivariance under Hamiltonian G -action

The crucial ingredient to make sense of integration is a concept of measure to integrate against. We formalize this as follows.

Definition. A choice of *fiberwise fundamental class* $[f]$ on a morphism

$$\begin{array}{c} \widehat{\mathbf{Fields}} \\ \downarrow f \\ \mathbf{Fields} \end{array}$$

in \mathbf{H} is (if it exists) a choice of dualizable $\tau \in \text{Mod}(\mathbf{Fields})$ together with a choice of equivalence

$$f_! f^*(1) \simeq f_* f^*(\tau).$$

Example. If we happen to have even a natural equivalence $f_! \simeq f_*$ then this induces in particular a fundamental class $[f]$ with vanishing twist, $\tau = 1$. This case of a single but two-sided (“ambidextrous”) adjoint to f^* is what is used to formulate integration in [HopkinsLurie14]. For our purposes we need the more general concept where only the above equivalence with possibly non-trivial τ is required.

Proposition. A choice of fiberwise fundamental class, induces a natural morphism of the form

$$[f]_A : A \otimes \tau \longrightarrow f_! f^* A.$$

Notice that this goes reverse to the $(f_! \dashv f^*)$ -unit.

Definition (path integral). Given an integral kernel

$$\begin{array}{ccccc} & & \mathbf{Fields}_{\text{traj}} & & \\ & \swarrow (-)|_{\text{out}} & & \searrow (-)|_{\text{in}} & \\ \mathbf{Fields}_{\text{out}} & & & & \mathbf{Fields}_{\text{in}} \\ & \swarrow \exp(\frac{i}{\hbar} S) & & \searrow & \\ & \mathbf{L}_{\text{out}} & & \mathbf{L}_{\text{in}} & \\ & & \mathbf{B}^n U(1) & & \\ & & \downarrow & & \\ & & \text{Line}(\ast) & & \end{array}$$

and a fiberwise fundamental class $[\text{in}]$, then the path integral is the composite

$$\mathbb{D} \int_{\mathbf{Fields}_{\text{traj}}} \exp(\frac{i}{\hbar} S) d\mu :=$$

$$\sum_{\mathbf{Fields}_{\text{out}}} \mathbf{L}_{\text{out}} \xleftarrow{\sum_{\mathbf{Fields}_{\text{out}}} \epsilon_{\mathbf{L}_{\text{out}}}} \sum_{\mathbf{Fields}_{\text{out}}} \text{out}! \text{out}^* \mathbf{L}_{\text{out}} \xleftarrow{\simeq} \sum_{\mathbf{Fields}_{\text{traj}}} \text{out}^* \mathbf{L}_{\text{out}} \xleftarrow{\sum_{\mathbf{Fields}_{\text{traj}}} \exp(\frac{i}{\hbar} S)} \sum_{\mathbf{Fields}_{\text{traj}}} \text{in}^* \mathbf{L}_{\text{in}} \xleftarrow{\simeq} \sum_{\mathbf{Fields}_{\text{in}}} \text{in}! \text{in}^* \mathbf{L}_{\text{in}} \xleftarrow{\sum_{\mathbf{Fields}_{\text{in}}} [\text{in}]} \sum_{\mathbf{Fields}_{\text{in}}} \mathbf{L}_{\text{in}} \otimes \tau$$

Example. If the linear homotopy-type theory is that given by vector spaces parameterized over (finite) sets, then this formula reproduces traditional matrix calculus in linear algebra.

Proposition. If the linear homotopy-type theory is that given by E -module bundles over (small) ∞ -groupoids for $E \in \mathbf{CRing}_{\infty}$ an E_{∞} -ring, then this formula reproduces the pull-push in twisted generalized E cohomology by twisted Umkehr maps as described in [ABG11].

1.3.5 Examples

Example a) particle at the boundary of 2d Poisson-Chern-Simons theory
(see example 7.3)

Example b) superstring at the boundary of the 3d Spin-Chern-Simons theory
(see example 7.4)

Example c) D-brane charge and T-duality
(see example 7.5)

1.3.6 Quantum anomaly cancellation via Cobounding field theory

Above we gave a definition of path integral that turns a correspondence given by field trajectories into a linear map between the spaces of quantum states of the incoming and the outgoing prequantum field theories. There are two obstructions to this construction being functorial

1. For measures with nontrivial τ the incoming space of quantum states picks up a twist and hence will not match the outgoing space of quantum states for two consecutive correspondences.
2. Even if the twist vanishes, then the choices of fundamental classes/measures might not be consistent in that doing the path integral first through one correspondence and then through the second is not equivalent to doing one single path integral through the composite correspondence.

These obstructions to functoriality prevent the path integral from actually constituting a quantum field theory. Such obstructions are known as quantum anomalies. Lifting them is “quantum anomaly cancellation”.

We close here by observing that the condition that the quantum anomalies in the path integral cancel is precisely the condition that there is a cobounding field theory in one dimension higher of which the given field theory is a boundary.

Take the functor on $\{\text{corner}\}\mathbf{Bord}_2$ to be given as follows:

- on objects: $X \mapsto \mathbf{Mod}(X) \in \mathbf{Mod}_2$
- on morphisms $(\xleftarrow{f} \xrightarrow{g}) \mapsto g!f^*$ (integral transform, as in Fourier-Mukai transf., Penrose transf., Harish-Chandra transf. etc.)
- on 2-morphisms

– of the form

$$\begin{array}{ccccc}
 & \text{Pic} & \xlongequal{\quad} & \text{Pic} & \\
 \uparrow \mathbf{L}_{\text{out}} & \nwarrow \exp(\frac{i}{\hbar} S) & & \nearrow & \uparrow \mathbf{L}_{\text{in}} \\
 \mathbf{Fields}_{\text{out}} & \xleftarrow{(-)|_{\text{out}}} & \mathbf{Fields}_{\text{traj}} & \xrightarrow{(-)|_{\text{in}}} & \mathbf{Fields}_{\text{in}}
 \end{array}$$

it assigns the induced

$$\text{out}^* \mathbf{L}_{\text{in}}^* \longrightarrow \text{in}^* \mathbf{L}_{\text{out}}^* \quad \text{in } \text{Mod}(\mathbf{Fields}_{\text{traj}})$$

• of the form

$$\begin{array}{ccccc}
 \mathbf{Fields}_{\text{out}} & \xleftarrow{(-)|_{\text{out}}} & \mathbf{Fields}_{\text{traj}} & \xrightarrow{(-)|_{\text{in}}} & \mathbf{Fields}_{\text{in}} \\
 \downarrow & & & & \downarrow \\
 * & \xlongequal{\quad} & & & *
 \end{array}$$

it assigns

$$\begin{array}{ccccc}
 \text{Mod}(\mathbf{Fields}_{\text{out}}) & \xleftarrow{\text{out}_! \text{in}^*} & \text{Mod}(\mathbf{Fields}_{\text{in}}) & & \\
 \downarrow \sum & \nwarrow \sum [\text{in}] & \nearrow \sum & & \downarrow \sum \\
 \mathbf{Fields}_{\text{out}} & & \mathbf{Fields}_{\text{in}} & & \mathbf{Fields}_{\text{in}} \\
 \downarrow & & & & \downarrow \\
 \text{Mod}(*) & \xlongequal{\quad} & & & \text{Mod}(*)
 \end{array}$$

Proposition. This assignment induces the above path integral. Therefore the path integral extends to a functor and is hence anomaly free if it arises as the boundary theory to the primary integral transform theory.

1.4 Summary (technical fact sheets)

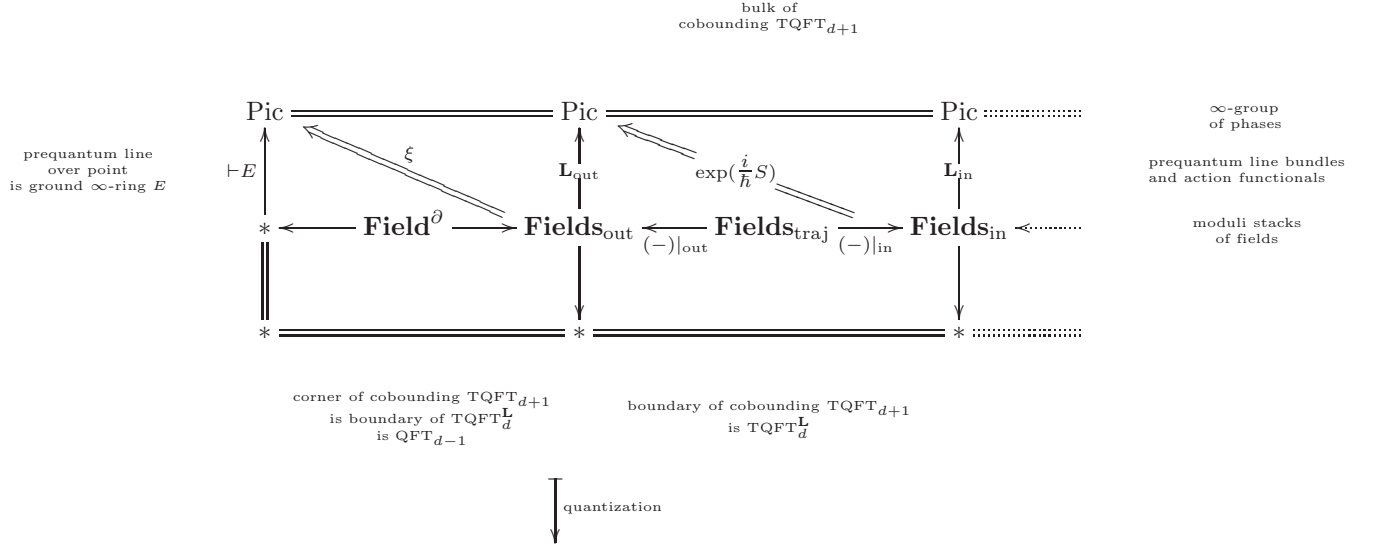
This document is an expanded set of notes for some talks I gave in February 2014. What follows here are some notes handed out in a talk that I gave together with Joost Nuiten in the String Geometry Network meeting at ESI in Vienna. This talk discussed more of the technical details of how pull-push in generalized cohomology is realized in the abstract theory and how it expresses cohomological quantization, following [Nuiten13].

1.4.1 Translation between linear homotopy-type theory, generalized cohomology and quantization

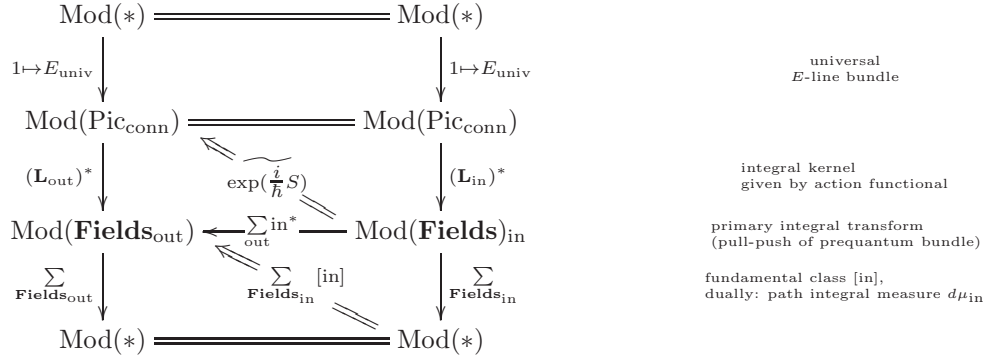
linear homotopy-type theory	twisted generalized cohomology	quantum theory
linear homotopy-type	(module-)spectrum	state space
multiplicative conjunction	smash product of spectra	composite system
dependent linear type	module spectrum bundle	
Frobenius reciprocity	six operation yoga in Wirthmüller context	linearity of integrals
dual type (linear negation)	Spanier-Whitehead duality	dual state space
invertible type	twist	prequantum line bundle, quantum anomaly
dependent sum	generalized homology spectrum	space of compactly supported quantum states “bra”
dual of dependent sum	generalized cohomology spectrum	space of quantum states “ket”
linear implication	bivariant cohomology	quantum operators
exponential modality	Goodwillie exponential	Fock space
dependent sum over finite homotopy type	Thom spectrum	
dualizable dependent sum over finite homotopy type	Atiyah duality between Thom spectrum and suspension spectrum	
(twisted) self-dual type	Poincaré duality	inner product (Hilbert) space
dependent sum coinciding with dependent product	ambidexterity, semiadditivity	system of inner product state spaces
dependent sum coinciding with dependent product up to invertible type	Wirthmüller isomorphism (twisted ambidexterity)	anomalous system of inner product state spaces
$(\sum_f \dashv f^*)$ -counit	pushforward in generalized homology	
(twisted-)self-duality-induced dagger of this counit	(twisted-)Umkehr map, fiber integration	quantum superposition and interference
linear polynomial functor	primary integral transform	propagator in cobounding TQFT _{d+1}
correspondence with linear implication	motive	prequantized Lagrangian correspondence, action functional
composite of this linear implication with unit and daggered counit	secondary integral transform	cohomological path integral, motivic transfer
trace	Euler characteristic	partition function

1.4.2 The quantization process.

Prequantum $d + 1$ -dimensional field theory in $\text{Corr}_2(\mathbf{H})$.



Quantum $d + 1$ -dimensional field theory in Mod_2 .



encodes

Quantum d -dimensional field theory

as unit-component in $\text{Mod}(*)$ of the above transformation: $\mathbb{D} \int_{\mathbf{Fields}_{\text{traj}}} \exp(\frac{i}{\hbar} S) d\mu :=$

$$\sum_{\mathbf{Fields}_{\text{out}}} \mathbf{L}_{\text{out}} \xleftarrow{\sum_{\mathbf{Fields}_{\text{out}}} \epsilon_{\mathbf{L}_{\text{out}}}} \sum_{\mathbf{Fields}_{\text{out}}} \text{out}! \text{out}^* \mathbf{L}_{\text{out}} \xleftarrow{\simeq} \sum_{\mathbf{Fields}_{\text{traj}}} \text{out}^* \mathbf{L}_{\text{out}} \xleftarrow{\sum_{\mathbf{Fields}_{\text{traj}}} \exp(\frac{i}{\hbar} S)} \sum_{\mathbf{Fields}_{\text{traj}}} \text{in}^* \mathbf{L}_{\text{in}} \xleftarrow{\simeq} \sum_{\mathbf{Fields}_{\text{in}}} \text{in}! \text{in}^* \mathbf{L}_{\text{in}} \xleftarrow{\sum_{\mathbf{Fields}_{\text{in}}} [\text{in}]} \sum_{\mathbf{Fields}_{\text{in}}} \mathbf{L}_{\text{in}} \otimes \tau$$

(secondary integral transform: pull-push of states).

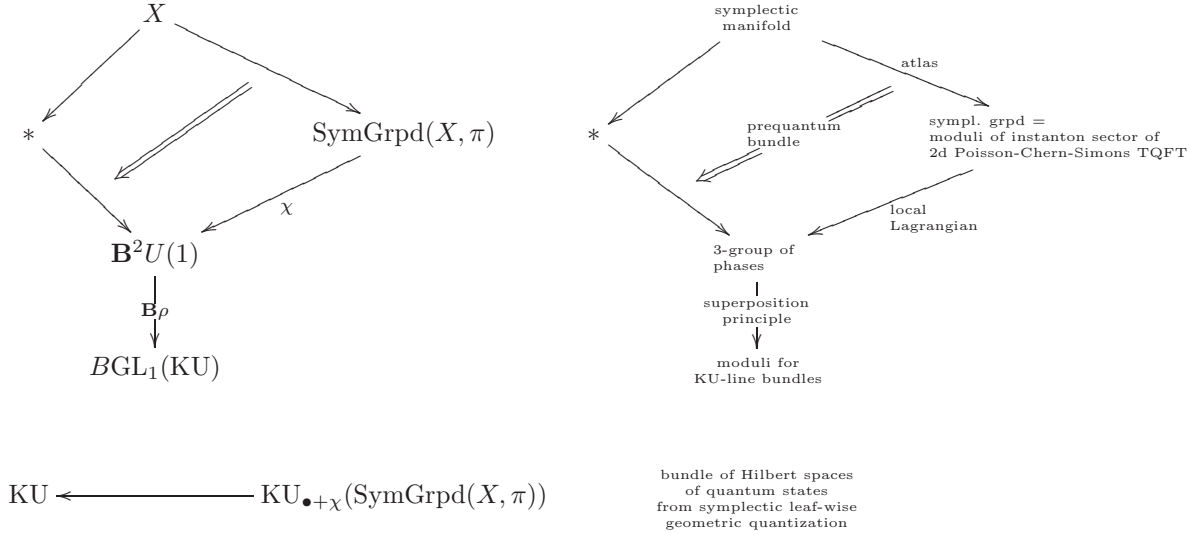
field theory	spaces of states	propagator
TQFT_{d+1}	$\text{Mod}_2 \in \text{Cat}_2$	integral transform
TQFT_d^τ	$\text{Mod}(*) \in \text{Mod}_2$	secondary integral transform, path integral
QFT_{d-1}	$\sum_X \mathbf{L}_X \in \text{Mod}(*)$	equivariance under Hamiltonian group action

1.4.3 Translation between linear homotopy-type theory in $E\text{Mod}$ and twisted E -cohomology.

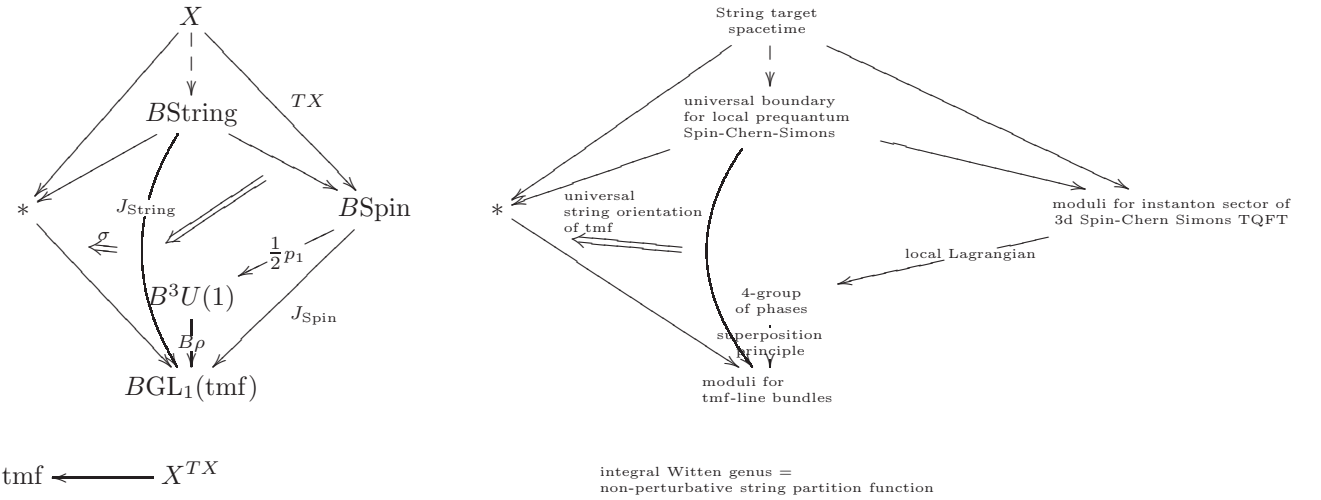
special case	linear homotopy-type theory	higher linear algebra viz. generalized cohomology theory
	$E \in \mathbf{CRing}_\infty$	ground ring
	$X \in \infty\mathbf{Grpd}$	base homotopy type (base space)
	$\tau : X \longrightarrow \mathbf{Pic}(E)$	twist
	$\widehat{\tau} := \tau^* \widehat{\mathbf{Pic}}(E) \in \mathbf{Mod}(X)$	E -line bundle
canonical twist on moduli for stable vector bundles	$J^E : \mathbb{Z} \times BO \xrightarrow{J} \mathbf{Pic}(\mathbb{S}) \xrightarrow{\mathbf{Pic}(\mathbb{S} \rightarrow E)} \mathbf{Pic}(E)$	J-homomorphism
	$\sum_X \widehat{\tau} \simeq E_{\bullet+\tau}(X)$	spectrum of τ -twisted E -homology cycles
trivial twist	$\sum_X 1_X \simeq E_{\bullet}(X) = E \wedge \Sigma_+^\infty X$	suspension spectrum
$X \xrightarrow{\xi} \mathbb{Z} \times BO$ modulating stable vector bundle	$\sum_X \widehat{J^E \circ \xi} = E \wedge X^\xi$	Thom spectrum
canonical twist on $X := BO\langle n \rangle$ $J_{BO\langle n \rangle}^E : BO\langle n \rangle \rightarrow BO \xrightarrow{J^E} \mathbf{Pic}(E)$	$\sum_{BO\langle n \rangle} \widehat{J_{BO\langle n \rangle}^E} \simeq MO\langle n \rangle$	universal Thom spectrum
low n	$n = 0: \quad MO$ $n = 1: \quad MSO$ $n = 2: \quad MSpin$ $n = 4: \quad MString$	$\left. \begin{array}{l} \text{Riemannian-} \\ \text{oriented-} \\ \text{spin-} \\ \text{string-} \end{array} \right\} \text{cobordism spectrum}$
	\mathbb{D}	Spanier-Whitehead duality
	$\mathbb{D} \sum_X \widehat{\tau} = E^{\bullet+\tau}(X)$	spectrum of τ -twisted E -cohomology cocycles
X compact smooth manifold with tangent bundle TX and stable normal bundle $NX = -TX$	$\mathbb{D}(E \wedge \Sigma_+^\infty X) \simeq E_{\bullet+NX}(X)$	Atiyah-Whitehead duality
	$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ & \swarrow \scriptstyle \tau_Z & \searrow \scriptstyle \tau_X \\ & \mathbf{Pic}(E) & \end{array}$	fiberwise E -orientation of τ_Z relative to τ_X
to the point	$\begin{array}{ccc} Z & \xrightarrow{\quad} & * \\ & \swarrow \scriptstyle \tau_Z & \searrow \scriptstyle 0 \\ & \mathbf{Pic}(E) & \end{array}$	τ_Z -twisted E -orientation of Z
vanishing twist on domain	$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ & \swarrow \scriptstyle 0 & \searrow \scriptstyle \tau_X \\ & \mathbf{Pic}(E) & \end{array}$	E -orientation of f
fiberwise fundamental class with twist τ	$f_! f^* \widehat{\tau}_X \simeq \mathbb{D} f_! f^* \mathbb{D}(\widehat{\tau}_X \otimes \widehat{\tau})$	fiberwise twisted Poincaré duality

1.4.4 Examples

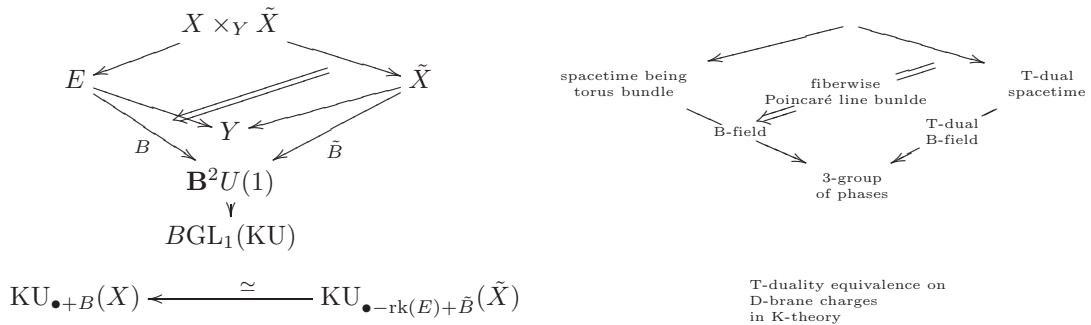
a) Particle at the boundary of 2d Poisson-Chern-Simons TQFT.



b) Superstring at boundary of 3d Spin-Chern-Simons TQFT.



c) D-Brane Charge and T-Duality.



2 Modal homotopy-type theory

We now discuss the theory in more detail. Here we begin by reviewing the ambient logical framework. We start in 2.1 with a lightning review of basics of (homotopy-)type theory. Then in 2.2 we state the simple but profound notion of modalities and opposite moments in type theory, following [Law91, Law94]. Finally in 2.3 we consider the pattern of such opposite moments which is called “cohesion” in [Law07] and the larger such pattern called “differential cohesion” in [Sch13a]. Then we review from [Sch13a] the structures induced by these axioms which we will show to serve as a formal foundation for quantization theory.

Of special interest for quantization is the linear structure that arises, in 2.3.4, from the opposite moments denoted $(\mathcal{R} \dashv \mathcal{J})$. This we turn to in 3 below.

We need to assume that the reader is familiar with basics of category theory, topos theory and preferably with basics of categorical logic, all explained for instance in [McLMoe92]. Our developments below take place in homotopy topos theory [Rezk10] also known as higher topos theory [Lurie06] and are naturally expressed in the internal language of higher toposes (∞ -toposes), which is essentially what is being called “homotopy type theory” [UFP13]. We will write “homotopy-type theory” here to indicate that we do not work with the formal syntax of [UFP13] but rather speak in terms of its semantics of geometric homotopy types (∞ -stacks) in the homotopy theory of ∞ -toposes. The type-theoretic perspective for us serves to make transparent the foundational and essentially formal nature of all assumptions and constructions such as to lend itself naturally to a fully formal syntactical formalization.

2.1 Types, Judgements and Deduction

While there is no room here to give a detailed introduction to homotopy toposes and homotopy type theory as their internal language (but see [Sch13a] for review of all the material that we need), the striking claim of [UFP13] is that at the bottom of it, homotopy type theory is much more elementary than it might seem, in fact that it is in a sense more elementary even than traditional elementary formal logic (since the homotopy-theoretic aspect is obtained not by adding axioms, but by omitting a traditional axiom, that of uniqueness of identity proofs). Therefore we can offer here a quick informal introduction from which the inclined reader might be able to get a working feeling for what is going on.

Imagine a computer terminal of the old sober type: a pitch black screen, empty except for a green prompt in the top left, expecting your input, like this:

⊢

The command line syntax is most elementary: you can enter expressions of the form

⊢ $x : X$

to be thought of as the judgement that a term x is of type X , a representative of some notion. For instance entering

⊢ $\frac{e^2}{4\pi\epsilon_0 G m_p m_e} : \text{RealNumber}.$

asks the system to judge that Dirac’s large number is a representative of the type of real numbers. The system will accept the input if the judgement is valid, and reject it otherwise.

First of all, for the above example to validate it must at least be true that “RealNumber” itself is a type known to the system. So there is a hierarchy of types Type of all types pre-installed and the system knows how to validate input of the form

⊢ RealNumber : Type

Once a judgement $X : \text{Type}$ is validated then the system may store that type. To indicate this it changes the appearance of the prompt to

$x : X \vdash$

Judgements entered after such a prompt are interpreted in this new context, hence may depend on the given type on the left. For instance after the system acknowledges the natural numbers, we can consider entering the judgement

$n: \text{NaturalNumber} \vdash \text{SU}(n): \text{Group}$

to express that for each natural number n , there is a group called $\text{SU}(n)$.

Every judgement in some context may be extended trivially to any other context and we assume that the system handles this *context extension* implicitly.

One dependent type pre-installed to the system is the collection of (gauge) equivalences between any two terms of the same type, which might be written

$X: \text{Type}; x_1, x_2: X \vdash (x_1 = x_2): \text{Type}$

expressing that for each type X and any two terms x_1, x_2 of that type, there is a type $(x_1 = x_2)$ of (gauge) equivalence between these.

This is to be such that a term of this type

$\vdash p: (x_1 = x_2)$

is a (gauge) equivalence between x_1 and x_2 .

For example if an X -dependent type $x: X \vdash P(x): \text{Type}$ is such that any two terms of it are either distinct or else equivalent in an essentially unique way, then it may be thought of as a *proposition* about the terms x of type X , where a term $p(x): P(x)$ is a proof of the proposition P about x . In this fashion the type theory here subsumes traditional formal logic; a simple but deep insight known variously as the *Brouwer-Heyting-Kolmogorov interpretation* or the *Howard isomorphism* or as the *propositions-as-types paradigm*, see section 1.11 of [UFP13].

However, crucial for the axiomatization of fundamental physics is that not all types are propositions in this sense, that type theory is genuinely more general than predicative logic, in that generally two terms of a type may be gauge equivalent in more than one essentially distinct ways.

Generally, given a dependent type

$x: X \vdash Y(x): \text{Type}$

such as

$n: \text{NaturalNumber} \vdash \mathbb{R}^n: \text{Type}$

then the system automatically deduces the existence of two further types, the sum (union) of all the types as x varies in X , called the *dependent sum type* and denoted

$\vdash \sum_{x: X} Y(x) : \text{Type}$

which is the type of terms that are in either of the $Y(x)$ as x varies in X , and the product of all these types, called the *dependent product* and denoted

$\vdash \prod_{x: X} Y(x) : \text{Type}$

which is the type of collections of terms in each $Y(x)$, as x varies in X .

For instance if for a given type X the system validates the existence of a term of the form

$\vdash p: \sum_{x_1: X} \prod_{x_2: X} (x_1 = x_2)$

then this means a choice of basepoint $x_1: X$ and a choice of X -parameterized equivalence of every other term x_2 of type X with x_1 .

Applied to an X -dependent type $P(x)$ which is a proposition in the above sense of the BHK interpretation, then the dependent sum reduces to the existential quantifier $\exists_{x \in X}$ in that a term of type $\sum_{x: X} P(x)$ is a proof

that there exists at least one x of type X such that $P(x)$ is true; and the dependent product \prod_X reduces to the universal quantifier $\forall_{x \in X}$ in that a term of type $\prod_{x: X} P(x)$ is a proof that $P(x)$ holds for all x parameterized over X .

On the other hand, applied to an X -dependent type Y whose dependence is trivial, then the dependent sum reproduces the categorical Cartesian product $X \times Y$ and the dependent product the function space $X \rightarrow Y$.

Imagine we could inspect the computer system's registers and memory. From this we deduce a category \mathbf{H} , called the *type system* or the *category of contexts*, whose objects are the given types in the plain context, and which for each X -dependent type Y has one morphism of the form $\left(\sum_{x: X} Y(x) \right) \xrightarrow{p_Y} X$ and which for each X -dependent term $x: X \vdash y(x): Y(x)$ has a commuting diagram of the form

$$\begin{array}{ccc} & \sum_{x: X} Y(x) & \\ y \nearrow & \downarrow p_Y & \\ X & \xlongequal{\quad} & X \end{array} \quad .$$

For every morphism $f: X \rightarrow Y$ in \mathbf{H} there is the pullback functor f^* between the slice categories and its left and right adjoints, respectively, are the above operations of dependent sum and dependent product

$$\left(\sum_f \dashv f^* \dashv \prod_f \right) : \mathbf{H}_{/X} \begin{array}{c} \xrightarrow{\sum_f} \\ \xleftarrow{f^*} \\ \xrightarrow{\prod_f} \end{array} \mathbf{H}_{/Y} \quad .$$

This key insight, which is the very basis of “categorical logic” and central to our development in 4 below, goes back to [Law69, Law70a].

One finds that the categories \mathbf{H} appearing this way are equivalently locally cartesian closed categories and one says that \mathbf{H} is the *categorical semantics* of the above formal syntax. More precisely, the above is the sketch of the construction of a 2-functorial equivalence

$$\begin{array}{ccc} \text{syntax} & & \text{semantics} \\ \text{MLTypeTheories} & \xleftarrow{\quad \simeq \quad} & \text{LccCategories} \end{array}$$

between a natural 2-category of (Martin-Löf intuitionistic) type theories and that of locally cartesian closed categories [See84, ClDy11]. In view of this equivalence we here speak about “type-semantics”.

Finally the homotopy theoretic aspect is reflected by interpreting the $(X \times X)$ -dependent type $(x_1 = x_2)$ of gauge equivalences between terms of type X as the diagonal morphism

$$\begin{array}{c} X \\ \downarrow \Delta_X \\ X \times X \end{array}$$

or rather as its fibrant resolution with respect to a homotopical fibration structure (essentially as in [Bro73], one of the earliest accounts of what today is called ∞ -topos theory), hence as a path space object X^I equipped with the two endpoint evaluation maps

$$\begin{array}{c} X^I \\ \downarrow (\text{ev}_0, \text{ev}_1) \\ X \times X \end{array} \quad .$$

Accordingly, homotopy type theory has semantics in locally Cartesian closed categories that are in addition equipped with a suitably compatibly homotopical fibration structure. These are such that they present locally cartesian closed ∞ -categories such as ∞ -toposes. For this see section 2 of [Shul12a].

2.2 Modalities, Moments and Opposites

A type system \mathbf{H} as specified in 2.1 provides a setting for types to be, but lacks as yet any determination of qualities these types may have, hence of modes of being. A central insight of traditional formal logic is, when generalized from propositions to types, that such *modalities* are formalized by monads on the type system [Law70a, Gol81, Mog91, Kob97, Shul12b], traditionally called *modalities* or *modal operators*:

Definition 2.1. A *modality* \bigcirc on a type system \mathbf{H} is a monad (an ∞ -monad) $\bigcirc : \mathbf{H} \rightarrow \mathbf{H}$. A *co-modality* \square is a co-monad (∞ -comonad) $\square : \mathbf{H} \rightarrow \mathbf{H}$. We say a \bigcirc -*modal type* (or \square -*co-modal type*) is a type equipped with the structure of a (co-)algebra over this monad.

(In practice we often suppress the “co-” terminologically, as it is determined by the context.)

Remark 2.2. The general theory of ∞ -monads on ∞ -categories is discussed in section 6.2 of [LurieAlg] and in [RiVe13]. By the homotopy monadicity theorem (theorem 6.2.2.5 of [LurieAlg] and def. 6.1.15 with section 7 of [RiVe13]) every ∞ -monad $\bigcirc : \mathbf{H} \rightarrow \mathbf{H}$ arises as the endomorphism monad $\bigcirc \simeq R \circ L$ of some ∞ -adjunction $(L \dashv R) : \mathbf{H} \leftrightarrow \mathcal{D}$ for some ∞ -category \mathcal{D} . By theorem 5.4.14 in [RiVe13] ∞ -adjunctions have the higher coherence data of their unit (and counit) uniquely (up to a contractible homotopy type of choices) induced from the underlying adjunction in the homotopy 2-categories. Therefore a choice of ∞ -adjunction $(L \dashv R)$ for \bigcirc re-encodes the coherence data of \bigcirc as a homotopy coherent monoid in the monoidal ∞ -category $\text{End}(\mathbf{H})$ equivalently as the choice of ∞ -category \mathcal{D} and the single datum of an ∞ -adjunction unit, see also remark 6.2.0.7 in [LurieAlg]. This allows to present ∞ -monads as ordinary monads on the homotopical fibration category underlying \mathbf{H} (as in 2.1 above), see [Hess10] for homotopy monadicity discussed in homotopical (model) categories this way. All ∞ -monads that we consider below arise as endomorphism monads of a given ∞ -adjunction.

If such a (co-)monad is *idempotent* in that applying it twice is equivalent to applying it just once, hence if it behaves as a projection, then it may be thought of as projecting out from any type one aspect or *moment* that it has. Traditionally this is called a *closure operator*:

Definition 2.3. A *moment* \bigcirc is an idempotent modality on \mathbf{H} , def. 2.1, a *co-moment* \square is an idempotent co-modality. Given a moment \bigcirc or co-moment \square write $\mathbf{H}_{\bigcirc}, \mathbf{H}_{\square} \hookrightarrow \mathbf{H}$ for the full subcategory of its modal types.

See [Shul12b] for moments in homotopy type theory. Here we focus on these idempotent monads; we encounter non-idempotent monads below in 2.3.4, in 4.2 and in 4. Notice that:

Proposition 2.4. For \square a moment (\bigcirc a co-moment), def. 2.3, then its modal types X , def.2.1, are equivalently those for which the unit $X \rightarrow \square X$ (the co-unit $\bigcirc X \rightarrow X$) is an equivalence. Moreover, these (co-)units exhibit the (co-)modal types as forming a reflective subcategory

$$\mathbf{H}_{\bigcirc} \overset{\hookrightarrow}{\underset{\dashv}{\rightleftarrows}} \mathbf{H} ,$$

resp. co-reflective subcategory

$$\mathbf{H}_{\square} \overset{\hookrightarrow}{\underset{\dashv}{\rightleftarrows}} \mathbf{H} .$$

It turns out to be specifically interesting to consider situations where a reflection and a co-reflection jointly exist in two different ways, either as an adjoint triple of the form

$$\mathbf{H}_{\bigcirc} \simeq \mathbf{H}_{\square} \overset{\overset{\bigcirc}{\hookrightarrow}}{\underset{\underset{\square}{\dashv}}{\overset{i_{\square} \simeq i_{\bigcirc}}{\rightleftarrows}}} \mathbf{H}$$

or of the form

$$\mathbf{H}_\circ \simeq \mathbf{H}_\square \begin{array}{c} \xrightarrow{i_\square} \\ \xleftarrow{\sim} \\ \xrightarrow{i_\circ} \end{array} \mathbf{H}$$

This is captured by the following

Definition 2.5. We say a moment \circ and co-moment \square are *dual* or *opposite* if they are adjoint

$$\circ \dashv \square \quad \text{or} \quad \square \dashv \circ$$

such that their categories of modal types are canonically equivalent.

Remark 2.6. The perspective of def. 2.5 has been highlighted in [Law91], where it is proposed (p. 7) that adjunctions of this form usefully formalize “many instances of the *Unity and Identity of Opposites*” that control Hegelian metaphysics [Heg12].

When we give such a duality a name D , we write

$$D : \circ \dashv \square \quad \text{or} \quad D : \square \dashv \circ$$

respectively. Given opposite moments $\circ \dashv \square$ or $\square \dashv \circ$, every type X sits naturally in a transformation

$$\square X \longrightarrow X \longrightarrow \circ X$$

between its two dual moments. This expresses how X is decomposed into these two moments. For stable homotopy types this further refines to the following much more detailed decomposition.

Definition 2.7. For pointed X we write $\tilde{\square}X$ for the fiber of $\square X \rightarrow X$ and $\tilde{\circ}X$ for the cofiber of $X \rightarrow \circ X$.

Proposition 2.8. *Given opposite moments $\circ \dashv \square$ as in def. 2.5, then every stable homotopy type X sits in a diagram of the form*

$$\begin{array}{ccccc} & \tilde{\circ}\Omega X & \xrightarrow{\quad} & \tilde{\square}\Sigma X & \\ & \nearrow & & \nwarrow & \\ \square\tilde{\circ}\Omega X & & X & & \circ\tilde{\square}\Sigma X \\ & \nwarrow & \nearrow & \nwarrow & \\ & \square X & \xrightarrow{\quad} & \circ X & \end{array},$$

and the two squares here are homotopy cartesian.

This was highlighted in [BNV13], see section 4.1.2 of [Sch13a].

Remark 2.9. The bottom piece of the diagram in prop. 2.8 is the basic decomposition of X into its dual moments $\square X$ and $\circ X$. The statement of prop. 2.8 is that if X is a stable homotopy type then first of all there is a further pair of opposite moments of X , namely $\tilde{\circ}\Omega X$ and $\tilde{\square}\Sigma X$, and second that X may be entirely reconstructed from either gluing $\circ X$ with $\tilde{\square}\Sigma X$ (along $\circ\tilde{\square}\Sigma X$) as well as from gluing $\square X$ with $\tilde{\circ}\Omega X$ (along $\square\tilde{\circ}\Omega X$).

There is also a decomposition into moments relative to a base type:

Proposition 2.10. *Let \bigcirc be a moment, def. 2.3, which preserves finite limits. Then for $X \in \mathbf{H}$ there is a moment \bigcirc_X on $\mathbf{H}_{/X}$ given by sending $(E \xrightarrow{p} X)$ to the left morphism in the pullback diagram*

$$\begin{array}{ccc} \bigcirc_X E & \longrightarrow & \bigcirc E \\ \downarrow & & \downarrow \bigcirc p \\ X & \longrightarrow & \bigcirc X \end{array},$$

where the bottom morphism is the \bigcirc -unit. Moreover, the universal factorization of p through this pullback

$$\begin{array}{ccccc} E & \dashrightarrow & \bigcirc_X(E) & \longrightarrow & \bigcirc E \\ & \searrow p & \downarrow & & \downarrow \bigcirc p \\ & & X & \longrightarrow & \bigcirc X \end{array}$$

is by a \bigcirc -equivalence $E \rightarrow \bigcirc_X E$, and this decomposition exhibits an orthogonal factorization system (\bigcirc -equivalences / \bigcirc_X -modal morphisms) in \mathbf{H} .

This is essentially observed in [CJKP97].

2.3 Axiomatic Metaphysics

We consider now on a type system as in 2.1 an iterative pattern of opposite moments, def. 2.5. By inspection we find that in the presence of such a pattern of dual moments the type system induces a natural formalization of large parts of pre-quantum and quantum physics.

2.3.1 Determination of the qualities of types

Notice a very basic example of opposite moments, def. 2.5:

Example 2.11. The dependent sum and dependent product over the empty type are adjoint

$$\sum_{\emptyset}(-) \dashv \prod_{\emptyset}(-).$$

This is equivalently the hom-adjunction of the empty type

$$((-) \times \emptyset) \dashv (\emptyset \rightarrow (-))$$

and this is equivalent to the adjunction between the co-monad constant on \emptyset and the monad constant on $*$

$$\emptyset \dashv *.$$

This adjunction exhibits dual moments. The canonical factorization

$$\emptyset \longrightarrow X \longrightarrow *$$

appearing in prop. 2.8 expresses any type X as intermediate between the empty type of no terms and the type of an entirely undetermined term.

Remark 2.12. On p. 7 of [Law91] the adjunction in example 2.11 is proposed to be a formalization, in the sense of remark 2.6, of what [Heg12] calls the unity of the moment of *Nichts* (nothing) and the moment of *reines Sein* (pure being) in *Werden* (becoming). It is in this sense that in [Law91] categories equipped with adjoint moments $\square \dashv \bigcirc$ are called “categories of being”. Indeed, according to [Heg12] I.1.1.C.a.4 “there is nothing which is not an intermediate state between being and nothing”, which one might feel is

well reflected by the fact that the canonical decomposition into moments in example 2.11 is the tautological $\emptyset \rightarrow X \rightarrow *$. But therefore this is a very unspecific hence un-determinate notion of “being”, and indeed this un-determination is what [Heg12] I.1.1.A takes as the characteristic of *reines Sein* (pure being). More determinate forms of being equipped with more qualities are to refine this (*Dasein*). We are hence led, following p. 7 of [Law91], to consider adding further dual moments in order to equip types with more determinate/more qualified being.

So consider then a further pair of opposite moments $b \dashv \#$ of the form $\square \dashv \bigcirc$,

$$\begin{array}{ccc} \emptyset & \subset & b \\ \downarrow & & \downarrow \\ * & \subset & \# \end{array}$$

Given this we may ask if these in turn are opposite themselves to pairs of opposites, in that we have

$$\begin{array}{ccc} f & \longrightarrow & b \\ \downarrow & & \downarrow \\ b & \longrightarrow & \# \end{array}$$

This yields

$$\begin{array}{ccc} & & f \\ & & \downarrow \\ \emptyset & \subset & b \\ \downarrow & & \downarrow \\ * & \subset & \# \end{array}.$$

We may consider incrementally adding further moments/qualities to the type system this way. Here we add one more stage of refinement:

Definition 2.13. Given a type system \mathbf{H} as in 2.1 (an ∞ -topos) a system of dual moments on \mathbf{H} , def. 2.5, of the form

$$\begin{array}{ccccccc} & & & & \mathfrak{R} & & \\ & & & & \downarrow & & \\ & & f & \subset & \mathcal{F} & & \\ & & \downarrow & & \downarrow & & \\ \emptyset & \subset & b & \subset & \mathfrak{S} & & \\ \downarrow & & \downarrow & & & & \\ * & \subset & \# & & & & \end{array}$$

we say is a structure of *differential cohesion* in \mathbf{H} [Sch13a]. Here “ \subset ” denotes inclusion of modal types.

Remark 2.14. Often we demand in addition that \int preserves finite products and that \mathfrak{R} preserves finite limits. Neither assumption is needed for the central statements about linear type theory below.

The system of moments in def. 2.13 determinates further “qualities” carried by all types. In the following we discuss consequences of these determinate qualities that play a role in the discussion of quantization below.

2.3.2 Realization of types as moduli stacks

We discuss here how the $(\oint \dashv \Im)$ -moments of def. 2.13 realize each type X as a higher Deligne-Mumford stack (higher Artin stack) in the sense of [Lurie09b]. This gives an explicit meaning to the statement that every type with the quality of def. 2.13 plays the role of a geometric moduli stack (for instance of higher gauge fields, this we discuss below in 2.3.3).

Claim 2.15. *Given differential cohesion, then for X any type then the \oint_X -modal X -dependent types form an étale topos over X and the \Im_X -modality equips this with an \mathbf{H} -structure sheaf in the sense of [Lurie09b]. Along \oint -modal morphisms these base change in a module-analog of dependent sum and dependent product.*

This is discussed in [Sch13a].

More in detail:

Proposition 2.16. *For \mathbf{H} equipped with differential cohesion and for $X \in \mathbf{H}$, the full subcategory*

$$\mathrm{Sh}(X) := (\mathbf{H}/_X)_{\oint_X} \begin{array}{c} \xleftarrow{\Re_X} \\ \xrightarrow{\Im_X} \end{array} \mathbf{H}/_X$$

on the \oint_X -modal types, prop. 2.10, is an ∞ -topos and equipped with the map

$$\mathcal{O}_X : \mathbf{H}_{\Re} \hookrightarrow \mathbf{H} \xrightarrow{X^*} \mathbf{H}/_X \xrightarrow{\Im} \mathrm{Sh}(X)$$

it is an \mathbf{H}_{\Re} -structured ∞ -topos in the sense of [Lurie09b] (with “admissible” morphisms in \mathbf{H} the \oint -modal morphisms).

We may call $\mathrm{Sh}(X)$ the *étale topos* of X .

Proposition 2.17. *For $f : Y \rightarrow X$ a morphism in \mathbf{H} which is \oint -modal as an X -dependent type, this canonically induces an étale morphism of structured ∞ -toposes, hence an étale geometric morphism of ∞ -toposes*

$$\mathrm{Sh}(Y) \simeq \mathrm{Sh}(X)/_Y \begin{array}{c} \xrightarrow{f_!} \\ \xleftarrow{f^*} \\ \xrightarrow{f_*} \end{array} \mathrm{Sh}(X)$$

together with an equivalence of structure sheaves

$$\mathcal{O}_Y \simeq f^* \mathcal{O}_X.$$

2.3.3 Gauge fields

We discuss here how the $(\int \dashv \flat)$ -moments of def. 2.13 exhibit each stable type as having the quality of a moduli stack of higher gauge fields/cocycles in generalized differential cohomology.

Claim 2.18. *Given differential cohesion, def. 2.13, then the decomposition of types into their $(\int \dashv \flat)$ -moments according to prop. 2.8 exhibits them as types of cocycles in generalized differential cohomology, hence as moduli for pre-quantum (“classical”) fields (higher gauge fields).*

This is discussed in [Sch13a, BNV13].

2.3.4 Infinitesimals, Linearity and Modules

We discuss how the $(\mathfrak{R} \dashv \mathfrak{f})$ -moments of def. 2.13 induce over each type X a closed symmetric monoidal category $(\text{Mod}(X), \otimes)$ which behaves like a category of bundles of modules over X , hence a generalization of a category of vector bundles over X .

Definition 2.19. Given a type system category \mathcal{C} with terminal object $*$ and coproducts, then its *maybe-modality* $*/$ is the modality (monad, def. 2.1) given by

$$*/ : X \mapsto X \sqcup *.$$

Here our notation reflects the following basic fact.

Proposition 2.20. *The maybe-modal types (def.2.1), hence the algebras in \mathcal{C} over the maybe-monad, def. 2.19, are equivalently the pointed objects; the category of algebras over the maybe monad is the co-slice $\mathcal{C}^{*/}$ under the point.*

Proof. That these algebras are the pointed objects is already equivalent to the statement of the unit axiom for algebras over the maybe-monad. Then action axiom is then automatically satisfied. \square

Definition 2.21. For $X \in \mathbf{H}$ a type, write $\mathbf{H}_{/X}^{X/} \simeq (\mathbf{H}_{/X})^{*/}$ for the category of algebras over the maybe-monad on the slice topos $\mathbf{H}_{/X}$, def. 2.19, hence by prop. 2.20 for the category of pointed objects in the slice.

Remark 2.22. An object in $\mathbf{H}_{/X}^{X/}$ may be interpreted a bundle over X which is equipped with a global section. The existence of such a global section is a property shared in particular by vector bundles and more generally by fiber bundles of modules, for which the global section is the zero-section.

We may axiomatically characterize those sectioned bundles over X which are like bundles of modules in that they are *linear* bundles as follows.

Definition 2.23. For X a type, an *infinitesimal extension* or *formal extension* of X is an object $(E \rightarrow X) \in \mathbf{H}_{/X}^{X/}$ such that the underlying morphism in \mathbf{H} is an \mathfrak{R} -equivalence, equivalently an \mathfrak{f} -equivalence. Write

$$\text{Mod}_{\mathfrak{R}}(X) \hookrightarrow \left(\mathbf{H}_{/X}^{X/} \right)^{\text{op}}$$

for the opposite of the full subcategory on these objects.

Remark 2.24. By prop. 2.10 $\text{Mod}_{\mathfrak{R}}(X)$ is in a sense dual to $\text{Sh}(X)$ of def. 2.16.

Remark 2.25. This definition depends on the system of dual moments in that by $(\mathfrak{R} \dashv \mathfrak{f} \dashv \mathfrak{S})$ the notion of “infinitesimal” in 2.23 is the same as the “infinitesimal” that characterizes the formally étale morphisms in def. 2.16.

Example 2.26. In the typical model of differential cohesion (as in section 4 of [Sch13a]), the objects E are given by sheaves over formal duals of commutative algebras (structured algebras) of the form $C^\infty(E) = C^\infty(X) \oplus N$, where N is a $C^\infty(X)$ -module and a nilpotent ideal in $C^\infty(E)$. If already $N \cdot N = 0$ then $C^\infty(E)$ is the square-0 extension of $C^\infty(X)$ induced from the module N . A morphism between two such

objects in $\text{Mod}(X)$ is a commuting diagram of algebras of the form

$$\begin{array}{ccc}
 & C^\infty(X) & \\
 \swarrow & & \searrow \\
 C^\infty(X) \oplus N_1 & \xrightarrow{\quad} & C^\infty(X) \oplus N_2 \\
 \searrow & & \swarrow \\
 & C^\infty(X) &
 \end{array}$$

where the vertical maps are the canonical inclusions and projections, respectively. For square-0 extensions this is equivalently a homomorphism of modules $N_1 \rightarrow N_2$.

In this fashion $\text{Mod}(X)$ faithfully contains a naive category of module bundles over X , but it may contain also richer module-like objects. In fact, the main theorem in [LurieForm] says that the infinitesimal extensions of X , def. 2.23, include the sheaves of L_∞ -algebras over X . This means that the “modules” considered here are to be thought of more generally as modules possibly equipped with further homotopy algebraic structure.

Lemma 2.27. *The inclusions $\text{Mod}(X)^{\text{op}} \hookrightarrow \mathbf{H}_{/X}^{X/}$ of def. 2.23 preserve limits and colimits.*

Proof. Limits in an undercategory are computed in the ambient category, and limits in an overcategory are computed as the limits of the diagram with a terminal object adjoined in the ambient category. Dually for colimits. We have to show that if the diagram in the under-overcategory is in the inclusion of the infinitesimal extensions, then so is its (co-)limit. Since \mathcal{f} preserves all these and by assumption on infinitesimal extension, applying \mathcal{f} to the diagrams with terminal (initial object) adjoined make them be diagrams of the shape an ∞ -groupoid with a terminal object, hence of a contractible ∞ -groupoid, hence be essentially constant on $\mathcal{f}(X)$. This shows that the limit of \mathcal{f} -local objects in the slice is itself an \mathcal{f} -equivalence. \square

We need the following general fact. Let \mathcal{C} be a closed symmetric monoidal category with finite limits and colimits and reflexive coequalizers. Write $*$ $\in \mathcal{C}$ for its terminal object and write $\mathcal{C}^{*/}$ for the category of maybe-modal types, def. 2.19, hence by prop. 2.20 of pointed objects, prop. 2.20, in \mathcal{C} . The maybe monad $*/$ is a commutative monoidal monad [Se12] and hence canonically induces the structure of a monoidal category on $\mathcal{C}^{*/}$.¹¹

Proposition 2.28. *The canonical tensor product induced on the maybe-modal types $\mathcal{C}^{*/}$ is the smash product “ \wedge ” of pointed objects. For $E_1, E_2 \in \mathcal{C}^{*/}$ this operation sends these to the following pushout of coproducts and tensor products formed in \mathcal{C}*

$$E_1 \wedge E_2 := * \coprod_{(E_1 \otimes *) \coprod (* \otimes E_2)} (E_1 \otimes E_2)$$

and this makes $(\mathcal{C}^{/}, \wedge, * \coprod *)$ a closed symmetric monoidal category for the internal hom of pointed morphisms.*

Proof. The canonically induced monoidal structure on the category of algebras of a commutative monad is often said to go back to [Ko70], where indeed the closed structure is discussed, from which the tensor may be obtained as the adjunct in suitable circumstances. The monoidal structure appears in print explicitly in [Se12] (section 2.2 and theorem 2.5.5). Inserting the maybe-monad into the coequalizer formula there straightforwardly yields the pushout diagram defining the smash product as it appears for instance in construction 4.19 and proposition 4.20 of [EM07]. \square

Since \mathcal{f} preserves limits and colimits and hence the construction of the smash product in prop. 2.28, the

¹¹ I am grateful to Todd Trimble and Mike Shulman for discussion of this point, and to Zhen Lin Low for pointing out the reference [Se12].

smash product in $\mathbf{H}_{/X}^{X/}$ induces a tensor product on infinitesimal extensions over X . On the other hand, the smash tensor unit $X \coprod X$ of $\mathbf{H}_{/X}^{X/}$ is not \mathcal{F} -local, hence is not an infinitesimal extension over X . Therefore we have:

Definition 2.29. For X a type, write $(\text{Mod}(X), \otimes)$ for the symmetric monoidal category without unit which is the opposite of the category of infinitesimal extensions of X , def. 2.23, equipped with the restriction of the smash product of prop. 2.28.

Example 2.30. Continuing example 2.26, consider for definiteness the case that these sheaves are representable, with $C^\infty(E_i) = C^\infty(X) \oplus N_i$ in the notation there. Then we have

$$\begin{aligned} C^\infty(E_1 \times_X E_2) &= (C^\infty(X) \oplus N_1) \coprod_{C^\infty(X)} (C^\infty(X) \oplus N_2) \\ &\simeq C^\infty(X) \oplus (N_1 \oplus N_2) \oplus (N_1 \otimes N_2) \end{aligned}$$

and

$$\begin{aligned} C^\infty(E_1 \coprod_X E_2) &= (C^\infty(X) \oplus N_1) \times_{C^\infty(X)} (C^\infty(X) \oplus N_2) \\ &\simeq C^\infty(X) \oplus (N_1 \oplus N_2) \end{aligned}$$

Therefore in these models the tensor product in def. 2.29 restricts to the ordinary tensor product of modules:

$$C^\infty(E_1 \wedge_X E_2) \simeq C^\infty(X) \oplus (N_1 \otimes N_2).$$

Remark 2.31. Example 2.30 shows that while the tensor unit of the smash product of pointed objects does not descend to a tensor unit of modules, in the standard models there is after all another object which is the tensor unit for $(\text{Mod}(X), \otimes)$. This is the incarnation of the structure sheaf \mathcal{O}_X in the present context, yields a closed symmetric monoidal category $(\text{Mod}(X), \otimes, \mathcal{O}_X)$.

Below in example 3.10 we find that forming categories of modules over types induces a “linear” (meaning: symmetric monoidal but not Cartesian monoidal) analog of the dependent type theory of 2.1. This linearity and its relation to quantization we discuss now in 3.

3 Linear homotopy-type theory

The two hallmarks of quantum physics are the *superposition principle* and *quantum interference*. Together they say that prequantum phases in $S^1 = U(1)$ are to be freely added, and then are to additively interfere in the ring $E = \mathbb{C}$ of complex numbers

$$U(1) \xrightarrow{\text{superposition}} \mathbb{Z}[U(1)] \xrightarrow{\text{interference}} \mathbb{C}.$$

This makes the outcome of quantization land in E -modules. For $E = \mathbb{C}$ these are complex vector spaces, but we will see that we should allow other choices of E , too. Notably for realizing quantum mechanics as the boundary field theory of the 2d Poisson-Chern-Simons TQFT we take $E = \text{KU}$ the complex K-theory ring (an E_∞ -ring), for which the superposition principle is given by *Snaith’s theorem* [Snaith79]:

$$BU(1) \xrightarrow{\text{superposition}} \mathbb{S}[BU(1)] \xrightarrow{\text{interference}} \mathbb{S}[BU(1)][\beta^{-1}] \simeq \text{KU}.$$

Generally, for E a commutative ring (or generally E_∞ -ring) write $E\text{Mod}$ for its category of modules. Under the tensor product \otimes_E of E -modules, this is a symmetric closed monoidal category. As such it is much like a cartesian closed category, only that the tensor product lacks a diagonal map. Therefore functions in $E\text{Mod}$ may depend only on single (linear) copies of their arguments, and for this reason [Gir87] called the internal logic of such non-cartesian closed symmetric monoidal categories *linear logic*.

3.1 Linear logic, Quantum logic, Linear type theory

Here we briefly survey the literature on quantum logic, and in fact quantum mechanics, realized in linear type theory; and then point out the need to refine this to dependent linear homotopy-type theory, which is what we discuss in the following sections.

While there are some deep technical theorems contained in this relation between linear logic, monoidal categories and quantum physics, namely the coherence theorems, we feel that beyond the technicalities there is a remarkable conceptual fact to take note of (which the following discussion hopefully conveys): pure logicians did secretly re-discover the basics of quantum mechanics (namely linear type theory) independently of physicists and from purely foundational reasoning. This is not unlike how, as discussed above in 2.1, pure logicians did also independently re-discover homotopy theory from purely foundational reasoning. Taken together this yields linear homotopy-type theory below in 3.2; and when we show that this naturally captures quantum field theory below in 7 then the natural origin of all this in the modern foundations of mathematics seems noteworthy.

The study of the internal type theory of non-Cartesian (closed) symmetric monoidal categories such as $E\text{Mod}$ above dates back at least to [Lam68], a systematic account following this is [Sza78]. After [Gir87] introduced “linear logic” this was soon realized [See89] to be the special case corresponding to those monoidal categories which are called star-autonomous. For instance the “proof nets” of linear logic are equivalently the string diagrams of these monoidal categories [Mel06]. Since the “linear” terminology is apt, today the internal logic/internal type theory of general symmetric monoidal categories is called linear logic, or rather *multiplicative intuitionistic linear logic* for definiteness, see for instance [HydP93, BPS94]. A recent survey of linear type theory is in [Mel09].

In parallel to this development, it was understood that *quantum logic* is naturally taken to be the internal linear logic of such monoidal categories/linear type theories [Ye90, Pr92, AbDu05, Du06]. When interpreted in the monoidal category of Hilbert spaces this reproduces the subobject lattices of the original quantum logic introduced in [BvN36], a fact explicitly noted in [Cro75, HeJa09]. Conversely, that therefore also categories of cobordisms with their natural symmetric monoidal structure, which by Atiyah-Segal serve as domains for quantum field theories (see [Lurie09a] for a review) interpret linear logic (hence quantum logic), was amplified in [Sl05, BaSt09].

In this quantum mechanical interpretation the proof nets of linear logic correspond to quantum circuits of quantum computing (see e.g. section 3.1 of [Kuper05]), and Girard’s operational semantics of linear logic via “Geometry of Interaction” (see def. 2.6 and remark 5.8 in [AHS02]) corresponds to the construction of categories of superoperators (see e.g. sections 1.4, 1.5 in [Kuper05]). That the additional abstract structure needed in order to restrict these to genuine *quantum operations* on spaces of mixed quantum states (and hence to provide the probabilistic meaning of quantum physics) is a “dagger-compact structure” on the linear type theory (this we come to in 4.5 below) was observed in [Sel07] following [AbCo04]. A quick survey of the wealth of constructions in quantum mechanics abstractly expressible this way is in [Coe12].

But we have to note that all this captures but a small fragment (to borrow that term) of quantum physics, namely just basic quantum mechanics, but not quantum field theory. Moreover, modern quantum field theory is characterized by three additional properties; it is

1. *Lagrangian*
2. *local*
3. *gauge*

quantum field theory, and we observe that interpreting each of these three items requires to refine the linear type theory of a single monoidal category, as considered in the above references, to dependent linear homotopy-type theory:

1. The quantum field theories of actual interest in nature and in theory are typically those that arise by a process of quantization from Lagrangian data. The quantization step is supposed to be essentially a

summing operation of linear types: a space of quantum states is a space of sections of a pre-quantum bundle, hence of “sums of fibers” of a prequantum bundle, and a quantum propagator between such states is a “path integral” summing up contributions of linear maps for all physical trajectories. This means that we need a linear analog of the dependent sum operation \sum discussed in 2.1, hence that we need dependent linear types to formalize quantization.

2. Locality in quantum field theory means, in the Schrödinger picture [Lurie09a], that spaces of states assigned to codimension-1 slices of spacetime/worldvolume arise by integrating up higher categorical data assigned to higher codimension strata. This means that spaces of states of gauge field theories are “higher linear types” (we focus on this below in 5).
3. Specifically, in the presence of non-trivial gauge equivalences, spaces of quantum states are no longer plain vector spaces, but are (co)chain complexes, called “BRST complexes” [HeTe92], whose differentials encode the gauge symmetry. This means spaces of states of gauge field theories are linear homotopy-types.

In view of all this and our above discussion in 2.1 we consider now the refinement of linear type theory to *linear homotopy-type theory* and hence in particular *dependent* linear type theory, whose induced logic is hence a form of predicate quantum logic. Or rather, as throughout, we stick to the categorical semantics of all this. A syntactic account of linear homotopy-type theory is being developed by Mike Shulman.

3.2 Linear homotopy-type theory

Given a cohesive space X , then there is also the category $E\text{Mod}(X)$ of E -module bundles over X . For instance if X is a physical phase space, then the prequantum line bundle is an invertible object in $E\text{Mod}(X)$. It follows that quantization is to take place in *dependent* linear type theory, parameterized over the cartesian types X of the pre-quantum geometry. According to Lawvere’s notion of categorical logic embodied in the notion of hyperdoctrines as made precise in [See83], this means, applied to linear logic, the following:

A *dependent linear logic* or *linear hyperdoctrine* is a category of contexts Γ , a symmetric closed monoidal category \mathcal{C}_Γ for each such context and functorially for each morphism of contexts $f : \Gamma_1 \rightarrow \Gamma_2$ an adjoint triple of functors

$$\left(\sum_f \dashv f^* \dashv \prod_f \right) : \mathcal{C}_{\Gamma_1} \begin{array}{c} \xrightarrow{-f_! \rightarrow} \\ \xleftarrow{f^*} \\ \xrightarrow{f_*} \end{array} \mathcal{C}_{\Gamma_2}$$

such that f^* is strong monoidal and satisfies Frobenius reciprocity, hence such that f^* is a strong closed monoidal functor. Typically one would also demand that consecutive such adjoint triples satisfy the Beck-Chevalley condition.

The categorical semantics for such dependent linear type theory has been studied in [Shul08, Shul12c]. But it is noteworthy that in just slightly different guise these axioms are much older: they are a version of Grothendieck’s “yoga of six functors” [May05], which were recognized as the abstract reason behind Verdier duality. Specifically, an adjoint triple $(f_! \dashv f^* \dashv f_*)$ with f^* strong closed monoidal is called a *Wirthmüller context* in [May05]. (The literature on Grothendieck’s six operations often considers (also) the dual *Grothendieck contexts*, e.g. [Pol08].)

We now state this more formally. We often say just “category” for “ ∞ -category”.

Definition 3.1. For \mathcal{C}, \mathcal{D} two closed symmetric monoidal categories, a *Wirthmüller context* $f : \mathcal{C} \rightarrow \mathcal{D}$ between them is a strong closed monoidal functor $f^* : \mathcal{D} \rightarrow \mathcal{C}$ such that it has a left adjoint and right adjoint $(f_! \dashv f^* \dashv f_*)$.

Often it is useful to equivalently reformulate closedness of f^* in terms of the following condition.

Definition 3.2. Given an adjunction $(f_! \dashv f^*)$ between symmetric monoidal categories such that f^* is a strong monoidal functor, then the condition that the canonical natural transformation

$$\overline{\pi} : f_!((f^*B) \otimes A) \longrightarrow B \otimes f_!(A)$$

is a natural equivalence is called the *projection formula*. The existence of the left adjoint $f_!$ and the validity of the projection formula is also referred to as *Frobenius reciprocity* in representation theory and in categorical logic (“hyperdoctrines”), and often just called *reciprocity*, for short.

Remark 3.3. Below in example 5.3 we observe that Frobenius reciprocity is a higher categorical linearity condition. This perspective is crucial for making transparent the nature of quantum anomaly cancellation that we come to below in 7.3, see remark 7.12 there.

A basic fact is that:

Proposition 3.4. *Given an adjoint pair $(f_! \dashv f^*)$ between closed monoidal categories with f^* a strong monoidal functor, then the condition that f^* is strong closed is equivalent to Frobenius reciprocity, def. 3.2, hence to $f_!$ satisfying its projection formula.*

Remark 3.5. If in a Wirthmüller context, def. 3.1, not only $f_!$ but also f_* satisfies its projection formula, then [Ha13] speaks of a “transfer context” (def. 4.9 there), because this turns out to be an abstract context in which Becker-Gottlieb transfer exists (prop. 4.14 there). The abstract construction of Becker-Gottlieb transfer is similar to the construction of Umkehr maps via fundamental classes in Wirthmüller contexts which we consider in 4.3 below.

The central concept of interest here is now the following.

Definition 3.6. A *model/semantics for linear homotopy-type theory* is a locally Cartesian closed ∞ -category \mathbf{H} (“of non-linear homotopy-types”) and a Cartesian fibration

$$\begin{array}{c} \mathbf{Mod} \\ \downarrow \\ \mathbf{H} \end{array}$$

(“of dependent linear homotopy-types”) such that the ∞ -functor

$$\mathbf{Mod} : \mathbf{H}^{\text{op}} \rightarrow \mathbf{Cat}_{\infty}$$

that classifies the fibration by the Grothendieck-Lurie construction ([Lurie06], section 3.2) takes values in Wirthmüller contexts, def. 3.1, hence sends objects $X \in \mathbf{H}$ to closed symmetric monoidal ∞ -categories $\mathbf{Mod}(X)$ and sends morphism $f : X \rightarrow Y$ to ∞ -functors $f^* : \mathbf{Mod}(Y) \rightarrow \mathbf{Mod}(X)$ which are strong monoidal, have a left and right adjoint, and are strong closed, hence, by prop. 3.4, satisfy Frobenius reciprocity.

Remark 3.7. Definition 3.6 is the evident ∞ -categorical version of the *closed monoidal fibrations* considered in [Shul08] (def. 13.1) and [Shul12c] (theorem 2.14). Mike Shulman is working on developing formal *syntax* for linear homotopy-type theory similar to the formal syntax for non-linear homotopy-type theory that is laid out in [UFP13]. This is to be such that def. 3.6 provides the corresponding ∞ -categorical semantics/models.

We consider now three classes of examples of semantics for linear homotopy-type theory.

Example 3.8. For \mathbf{H} an ∞ -topos and $f : X_1 \rightarrow X_2$ any morphism in \mathbf{H} , the induced étale geometric morphism on slice toposes is a cartesian Wirthmüller context between the slice toposes:

$$\left(\sum_f \dashv f^* \dashv \prod_f \right) : (\mathbf{H}_{/X_1}, \times_{X_1}, X_1) \longrightarrow (\mathbf{H}_{/X_2}, \times_{X_2}, X_2).$$

Hence the self-indexing

$$\begin{array}{c} \mathbf{H}^{\Delta^1} \\ \downarrow \text{cod} \\ \mathbf{H} \end{array}$$

of an ∞ -topos \mathbf{H} (the codomain fibration, see [Lurie06] 2.4.7) is a model for linear homotopy-type theory, def. 3.6.

Proof. The left adjoint $f_! = \sum_f$ (dependent sum) sends slice objects $(A \rightarrow X_1)$ to the composite $(A \rightarrow X_1 \xrightarrow{f} X_2)$. Therefore by prop 3.4 it is sufficient to exhibit Frobenius reciprocity in the form

$$A \times_{X_1} f^* B \simeq A \times_{X_2} B.$$

But this is equivalently the pasting law for pullbacks in \mathbf{H} :

$$\begin{array}{ccccc} A \times_{X_1} f^* B & \longrightarrow & f^* B & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & X_1 & \xrightarrow{f} & X_2 \end{array}$$

□

Example 3.9. For \mathbf{H} a topos and $X \in \mathbf{H}$ an object, write $\mathbf{H}_{/X}^{X/} \simeq (\mathbf{H}_X)^{*/}$ for the category of pointed objects in the slice topos over X , prop. 2.20. By prop. 2.28 this is a closed symmetric monoidal category $(\mathbf{H}_{/X}^{X/}, \wedge_X, X \amalg X)$ under the smash product \wedge_X . For $f : X \rightarrow Y$ a morphism in \mathbf{H} , the base change inverse image of example 3.8 lifts to a functor $f^* : \mathbf{H}_{/Y}^{Y/} \rightarrow \mathbf{H}_{/X}^{X/}$. This is strong closed monoidal with respect to the smash product structure and has a left and right adjoint, hence constitutes a Wirthmüller context

$$\left(\sum_f \dashv f^* \dashv \prod_f \right) : (\mathbf{H}_{/X}^{X/}, \wedge_X, X \amalg X) \rightarrow (\mathbf{H}_{/Y}^{Y/}, \wedge_Y, Y \amalg Y).$$

This appears as examples 12.13 and 13.7 in [Shul08] and example 2.33 in [Shul12c].

Proof. For $f : X \rightarrow Y$ any morphism in \mathbf{H} then the base change inverse image $f^* : \mathbf{H}_{/Y}^{Y/} \rightarrow \mathbf{H}_{/X}^{X/}$ preserves pointedness, and the pushout functor $f_! : \mathbf{H}_{/X}^{X/} \rightarrow \mathbf{H}_{/Y}^{Y/}$ preserves co-pointedness. These two functors hence form an adjoint pair $(f_! \dashv f^*) : \mathbf{H}_{/X}^{X/} \rightarrow \mathbf{H}_{/Y}^{Y/}$. Moreover, since colimits in the under-over category $\mathbf{H}_{/X}^{X/}$ are computed as colimits in \mathbf{H} of diagrams with an initial object adjoined, and since by the Giraud axioms in the topos \mathbf{H} pullback preserves these colimits, it follows that $f^* : \mathbf{H}_{/Y}^{Y/} \rightarrow \mathbf{H}_{/X}^{X/}$ preserves colimits. Since by prop. 2.20 $\mathbf{H}_{/X}^{X/}$ is presentably monadic over $\mathbf{H}_{/X}$ (via the maybe-monad $* \amalg (-)$) we have by 2.78 in [AR94] that $\mathbf{H}_{/X}^{X/}$ and $\mathbf{H}_{/Y}^{Y/}$ are locally presentable categories, so that by the adjoint functor theorem it follows that f^* has also a right adjoint $f_* : \mathbf{H}_{/X}^{X/} \rightarrow \mathbf{H}_{/Y}^{Y/}$.

To see that f^* is a strong monoidal functor observe that the smash product is, by prop. 2.28, given by a pushout over coproducts and products in the slice topos. As above these are all preserved by pullback. Finally to see that f^* is also a strong closed functor observe that the internal hom on pointed objects is a fiber product of cartesian internal ohms. These are preserved by example 3.8, and the fiber product is preserved since f^* preserves all limits. Hence f^* preserves also the internal homs of pointed objects. □

Example 3.10. Given an ∞ -topos \mathbf{H} equipped with moments $(\mathfrak{R} \dashv \oint)$, def. 2.13, then for $f : X \longrightarrow Y$ a morphism in \mathbf{H} the Wirthmüller context on pointed objects of def. 3.9 restricts to infinitesimal extensions, def. 2.23,

$$\left(\sum_f \dashv f^* \dashv \prod_f \right) : \mathrm{Mod}_{\mathfrak{R}}(X)^{\mathrm{op}} \longrightarrow \mathrm{Mod}_{\mathfrak{R}}(Y)^{\mathrm{op}} .$$

Hence

$$\begin{array}{c} \mathrm{Mod}_{\mathfrak{R}} \\ \downarrow \\ \mathbf{H} \end{array}$$

is a model for linear homotopy-type theory, def. 3.6.

Proof. By lemma 2.27 it follows that $f_!$ and f^* preserve infinitesimal extensions and that the restriction of f^* still preserves colimits. Therefore to see that f_* restricts it is sufficient to see that $\mathrm{Mod}(X)^{\mathrm{op}}$ is locally presentable. Since it is the essential fiber of $\oint_X : \mathbf{H}_{/X}^{X/} \rightarrow \mathbf{H}_{/\oint(X)}$ over the singleton subcategory on $\mathrm{id}_{\oint(X)}$ this statement follows by corollary A.2.6.5 in [Lurie06]. \square

In applications already a small subclass of the linear homotopy-types in example 3.10 carry a fair amount of structure: over sites consisting of formal duals of square-0 extensions of commutative (E_{∞}) -rings E , E -modules $N \in E\mathrm{Mod}$ are modules in the above abstract sense, by example 2.26, and often we want to consider only these “representable” modules. The following example shows that at least over geometrically discrete homotopy-types, these representable modules in themselves constitute a linear homotopy-type theory

Example 3.11. Let \mathcal{V} be a closed symmetric monoidal ∞ -category with all small ∞ -limits and ∞ -colimits, such as $\mathcal{V} = E\mathrm{Mod}$ for $E \in \mathrm{CRing}_{\infty}$. For X a groupoid, write

$$\mathcal{V}(X) := \mathrm{Func}(X, \mathcal{V})$$

for the ∞ -category of ∞ -functors $X \rightarrow \mathcal{V}$ (also called \mathcal{V} -local systems on the homotopy type X). For $f : X \longrightarrow Y$ a morphism of ∞ -groupoids, the pullback (precomposition) ∞ -functor $f^* : \mathcal{V}(Y) \rightarrow \mathcal{V}(X)$ has a left and right ∞ -adjoint $f_!$ and f_* , given by left and right ∞ -Kan extension ([Lurie06] 4.3), hence constitutes an adjoint triple

$$\left(\sum_f \dashv f^* \dashv \prod_f \right) : \mathcal{V}(X) \begin{array}{c} \xrightarrow{f_!} \\ \xleftarrow{f^*} \\ \xrightarrow{f_*} \end{array} \mathcal{V}(Y) .$$

These are Wirthmüller contexts and hence make

$$\begin{array}{c} \mathcal{V}(-) \\ \downarrow \\ \mathrm{Grpd}_{\infty} \end{array}$$

a model for linear homotopy-type theory, def. 3.6.

Proof. That $f_!$ and f_* are given by ∞ -Kan extension is prop. 4.3.3.7 in [Lurie06]. We need to show that f^* is strong closed, hence by prop. 3.4 that $(f_! \dashv f^*)$ satisfies Frobenius reciprocity. To that end notice that by the very definitions 4.3.2.2 and 4.3.3.2 in [Lurie06] to which prop. 4.3.3.7 there appeals, ∞ -Kan is given *pointwise* at $y \in Y$ given by ∞ -colimit over the homotopy fiber $f^{-1}(y) \hookrightarrow X$:

$$(f_! A)(y) \simeq \lim_{\longrightarrow_{x \in f^{-1}(y)}} A(x) .$$

(For X and Y just 1-groupoids and \mathcal{V} locally presentable this follows, with [Lurie06] A.3.3, also from the more traditional fact that homotopy Kan extension is pointwise/strong [Ci03].) Hence for $f : X \rightarrow Y$, and for $A, B \in \mathcal{V}(Y)$ we have naturally in $x \in X$ the equivalences

$$\begin{aligned} f_!(f^*B) \otimes A &= \lim_{\longrightarrow x \in f^{-1}(y)} (f^*B(x) \otimes A(x)) \\ &\simeq \lim_{\longrightarrow x \in f^{-1}(y)} (B(y) \otimes A(x)) \\ &\simeq B(y) \otimes \lim_{\longrightarrow x \in f^{-1}(y)} A(x) \\ &= (B \otimes f_!A)(x) \end{aligned}$$

□

Remark 3.12. Below we often consider a model for linear homotopy-type theory $\text{Mod}(-)$ as in example 3.11 together with a differentially cohesive ∞ -topos \mathbf{H} , def. 2.13. Then for $\mathbf{Fields} \in \mathbf{H}$ an object we write

$$\text{Mod}(\mathbf{Fields}) := \text{Mod}\left(\int \mathbf{Fields}\right)$$

for the ∞ -category of linear homotopy-types over $\int X \in \infty\text{Grpd}$ produced by the shape modality (the “geometric realization” of the moduli stack \mathbf{Fields}).

For reference notice that example 3.11 reduces to the following special cases.

Example 3.13. For $E = \mathbb{S}$ the sphere spectrum, then $E\text{Mod}(-) = \mathbb{S}\text{Mod}(-) \simeq \text{Spectra}(-)$ is the theory of parameterized spectra. This was shown to be a model for linear homotopy-type theory in our sense here in [MaSi06] and [ABGHR08] (under the translation in [ABG10]).

Example 3.14. For \mathcal{V} a closed monoidal category with all small limits and colimits, such as $\mathcal{V} = E\text{Mod}$ for $E \in \text{CRing}$, then the functor

$$[-, \mathcal{V}] : \text{Set}^{\text{op}} \longrightarrow \text{MonCat}$$

takes values in Wirthmüller morphisms.

This appears as example 2.2, 2.17 in [Shul12c].

Remark 3.15. Examples 3.9, 3.10 and 3.11 are linearized variants of example 3.8. Accordingly we may think of the notion of systems of Wirthmüller morphisms as being a generalization of categorical logic (“hy-perdoctrines”) to non-cartesian and in particular to linear contexts. Notice that in a cartesian Wirthmüller context duality becomes trivial, in that $\mathbb{D}A \simeq 1$ for all objects A . To the extent that all the key constructions that we consider here in the following involve duality, they are interesting the “less cartesian” (“less classical”) the ambient Wirthmüller context is.

We now consider some basic constructions in dependent linear type theory/Wirthmüller contexts. The key notion for the discussion to follow is def. 4.8 below, which gives an abstract formalization of a notion of *measure* against which one can perform an abstract kind of integration.

3.3 Continuation modality and Dependent linear De Morgan duality

In general linear logic there is no notion of negation, but its role is played by duality. Just as negation intertwines ordinary logical conjunction and disjunction, a basic fact called *de Morgan duality*, so duality in linear logic intertwines linear dependent sum with linear dependent product. This is prop. 3.18 below. Since it is useful to freely pass back and forth along this linear de Morgan duality, we here collect some basic constructions and facts.

Definition 3.16. For $(\mathcal{C}, \otimes, [-, -], 1)$ a closed symmetric monoidal category, write

$$\mathbb{D} := [-, 1] : \mathcal{C}^{\text{op}} \longrightarrow \mathcal{C}$$

for the weak dualization functor.

Remark 3.17. As usual, we say an object $X \in \mathcal{C}$ is dualizable if it has a (“strong”) dual with respect to the tensor product \otimes , and we say X is *invertible* if the unit and counit of the duality map are equivalences. Generally $\mathbb{D}X$ is usually called the *weak* dual of X . There is a canonical natural morphism

$$X \longrightarrow \mathbb{D}\mathbb{D}X$$

which exhibits the unit of the modality (monad) $\mathbb{D}^2(-)$, also called the *continuation monad* [Mel08].

Proposition 3.18. *For f a Wirthmüller morphism, def. 3.1, the left and right adjoints are intertwined by weak duality in that there is a natural equivalence*

$$f_* \circ \mathbb{D} \simeq \mathbb{D} \circ f_!$$

hence

$$\prod_f \circ \mathbb{D} \simeq \mathbb{D} \circ \sum_f .$$

This is a special case of a more general consequence of the axioms of Wirthmüller morphisms which appears in [May05] as prop. 2.8, prop. 2.11. (In fact the “sixth operation”, the internal hom $[-, -]$, for us here only ever appears in its specialization to weak duality $\mathbb{D} = [-, 1]$.)

Remark 3.19. Prop. 3.18 is an incarnation of *de Morgan duality* in linear logic. This involves in particular that if $A \in \mathcal{D}$ is dualizable, then the $(f^* \dashv f_*)$ -unit η^{f^*} on A is the dual of the $(f_! \dashv f^*)$ -counit $\epsilon^{f_!}$ on the dual of A .

$$\eta_A^{f^*} \simeq \mathbb{D}(\epsilon_{\mathbb{D}A}^{f_!}) .$$

3.4 Linear polynomial functors and Primary integral transforms

A central notion in type-semantics is the following:

Definition 3.20. Given a model of linear homotopy-type theory $\text{Mod} \rightarrow \mathbf{H}$, def. 3.6, then a *multivariate polynomial functor* $P : \text{Mod}(X_1) \rightarrow \text{Mod}(X_2)$ is a functor of the form

$$P \simeq \sum_{f_2} \prod_g f_1^*$$

for a given diagram in \mathbf{H} the form

$$\begin{array}{ccc} & Y & \xrightarrow{g} Z \\ f_1 \swarrow & & \searrow f_2 \\ X_1 & & X_2 \end{array} .$$

If here $g \simeq \text{id}$, hence if the diagram is a *correspondence*

$$\begin{array}{ccc} & Z & \\ f_1 \swarrow & & \searrow f_2 \\ X_1 & & X_2 \end{array} ,$$

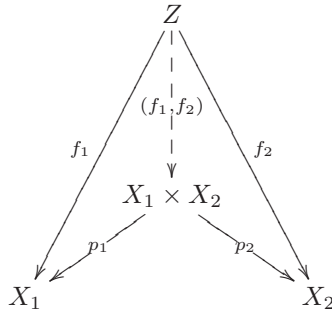
then its polynomial functor $\sum_{f_2} f_1^*$ is called a *linear polynomial endofunctor*.

Remark 3.21. In the existing type-theoretic literature the focus is on polynomial *endofunctors*, since the initial algebras over such endofunctors embody a useful notion of inductive types (“W-types”). Polynomial endofunctors in non-linear homotopy-type theory (our example 3.8) have been considered in [Ko12, vdBMo13].

Remark 3.22. In the existing representation-theoretic literature, linear polynomial functors are known as *categorified integral transforms*, as for instance in “Fourier-Mukai transform”, “Penrose transform”, “Harish-Chandra transform”, see e.g. [BZFN08, BZN09]. Here we will call these *primary* integral transforms for emphasis, since below in 4.4 our focus is on another concept of “secondary” integral transform that will turn out to be “boundaries” for the primary transforms.

Categorified integral transforms are often understood to have as correspondence space the product space (fiber-product in the relative case) and given there not just by pull-push but by pull-tensor-push. This relates to the above via the following basic fact (see for instance also p. 10 of [BZN09]).

Proposition 3.23. *Given a correspondence and its universal factorization through the product-space correspondence*



then pull-push through the given correspondence is equivalent pull-tensor-push through the product-correspondence, for integral kernel given by $K := \sum_{(f_1, f_2)} 1_Z$:

$$\sum_{f_2} \circ f_1^* \simeq \sum_{p_2} \circ (K \otimes (-)) \circ p_1^*.$$

Proof. By Frobenius reciprocity, def. 3.2:

$$\begin{aligned} (f_2)_!(f_1)^* A &\simeq (p_2)_!(f_1, f_2)_!(f_1, f_2)^* p_1^* A \\ &\simeq (p_2)_!(f_1, f_2)_!(((f_1, f_2)^*(p_1^* A)) \otimes 1_Z) \\ &\simeq (p_2)_!(p_1^* A \otimes ((f_1, f_2)_! 1_Z)) \\ &=: (p_2)_!(p_1^* A \otimes K). \end{aligned}$$

□

The terminology “polynomial functor” in def. 3.20 is motivated from the following basic example.

Example 3.24. Consider **H** the topos of sets and $\text{Set}^{\Delta^1 \text{ cod}} \rightarrow \text{Set}$ the associated dependent type structure, example 3.8. If we think of finite sets under their cardinality as representing natural numbers, then a polynomial functor with $X \simeq *$ the singleton acts indeed as a polynomial function under cardinality, summing up powers as given by the cardinalities of the fibers of g . In the other extreme, for general X but with $g = \text{id}$ then a polynomial functor is analogously given by multiplication by a matrix with entries in natural numbers.

Similarly for $\text{Vect}(-) \rightarrow \text{Set}$ the linear type theory of vector spaces over sets, example 3.14, then for finite dimensional vector spaces over finite sets, a polynomial functor acts as a polynomial function on the *dimensions* of these vector spaces.

Remark 3.25. Example 3.24 shows that the concept of polynomial functor is a *categorification* of that of polynomial function, hence a kind of “higher dimensional” polynomial function.

In parallel to this, below in 7.3 we find that linear polynomial functors in linear homotopy-type theory constitute the propagators of $(d + 1)$ -dimensional topological quantum field theories between “categorified” spaces of states, and that they encode (the quantum anomaly cancellation of) d -dimensional topological quantum field theories with propagators acting between uncategorified spaces of states. Further discussion of this phenomenon of similar structures appearing in different dimensions we give below in 8.1.

4 Computational homotopy-type theory

We had seen that a modality \bigcirc (a monad on the type system, def. 2.1), which is idempotent in that the monad operation $\bigcirc\bigcirc \rightarrow \bigcirc$ is an equivalence, serves to encode qualities and “moments” of types, def. 2.3. If however the monad is not idempotent then its successive applications equip types with successive operations, hence with an “evolution” or “computation”. That indeed logically this captures computation with side effects in functional programming is the celebrated insight of [Mog91, Kob97]. Accordingly one sometimes refers to type systems equipped with a (non-idempotent) monad \bigcirc as “computational type theory” [BBdP93, Mend02].¹²

On the other hand, in the Schrödinger picture of quantum physics, evolution is controlled by linear functionals that arise under quantization as (secondary) s induced by integral kernels. We now discuss a general abstract formalization of such integral kernels and of their integral transformations in computational homotopy-type theory. The non-idempotent monads that appear are those induced by adjoint triples of linear homotopy-types as in 3.

Taken together, this means that the following may also be thought of as formalizing aspects of quantum computation. Indeed, there is a vague similarity to the proposal in [LaFa12] of identifying the language for quantum computation to be (linear) type theory equipped with a suitable monad.

4.1 Motivation

To motivate our abstract formalization to follow, it is helpful to notice that the operation of pull-push of quasi-coherent sheaves of modules through correspondences famously embodies a higher analog of integral transformations via integral kernels [Pol08, BZFN08]; the seminal historical example being the Fourier-Mukai transform. On the other hand, another higher analog of integral transform theory is given by pull-push in generalized twisted cohomology (twisted Umkehr maps [ABG11]), as it appears for instance in the construction of string topology operations (see example 7.1 below) and in the discussion of T-duality. What has apparently been lacking is the identification of a core general abstract framework for integral transforms that would make both these applications be special cases in a way that their main properties follow formally from general statements. We will show that in a precise sense the latter notion is a “boundary theory” of the former, hence may be called a “secondary” integral transform in analogy with the concept of secondary characteristic classes.

At least a central ingredient of such a formalization should be a “yoga of six operations” that Grothendieck famously identified as the abstract mechanism underlying Grothendieck-Verdier duality. A clean analysis of the possible flavors of this six-operations axiomatics and their corresponding implications has been given in [May05]. The central formal consequence highlighted there is the “Wirthmüller isomorphism” which says, when it applies, that pull-push respects dualization, up to, possibly, a certain natural twist.

¹² In the introduction of [BBdP93] it is observed that this type-theoretic interpretation of “modality” in fact nicely harmonizes with the default interpretation of modal logic as being about the modality T of “possibility”:

The starting point for Moggi’s work is an explicit semantic distinction between *computations* and *values*. If A is an object which interprets the values of a particular type, then $T(A)$ is the object which models computation of that type A . [...] For a wide variety of notions of computation, the unary operator $T(-)$ turns out to have the categorical structure of a *strong monad* on an underlying cartesian closed category of values. [...] On a purely intuitive level and particularly if one thinks about non-termination, there is certainly something appealing about the idea that a computation of type A represents the possibility of a value of type A .

We observe below, in example 6.2 following [Nuiten13], that it is precisely this statement which also controls the twisted Umkehr maps in generalized cohomology theory [ABG11], using that these Umkehr maps are built by first dualizing (Poincaré duality), then pushing, then dualizing back, possibly picking up a twist thereby.

Motivated by this example, in this note we are after identifying a general abstract (higher) algebraic formalization of (secondary) integral kernels and integral transformations.

Specifically we are after a formalization that may mesh well with axiomatic cohesion. In the context of axiomatic cohesion one regards the ambient category of spaces and (principal) bundles over them as forming an indexed cartesian closed monoidal category equipped with a system (a “yoga”) of base change morphisms (hyperdoctrine) and equipped with some monads used to express the intended modalities characterizing such spaces. Below it turns out that we are working essentially in a linear monoidal (“tensor”) or “quantum” variant of this setup, where instead of an indexed topos we have an indexed closed monoidal linear category (as in [Shul12c], theorem 2.14). Analogous to axiomatic cohesion we find that the yoga of adjoint base change functors becomes most natural and transparent when expressed in terms of (just) the (co-)monads induced by these adjunctions.

4.2 Exponential modality, Linear spaces of states and Fock space

The secondary integral transforms that we discuss below in 4.4 act on what in the typical model of linear homotopy-type theory are linear spaces of sections of bundles of modules. In the application to quantization in 7 these are going to be thought of as spaces of quantum states. Here we discuss how the concept of forming spaces of sections is naturally captured by the *exponential modality* of linear logic, lifted to linear homotopy-type theory.

The full set of axioms for linear logic as introduced in [Gir87] contains – on top of the “multiplicative fragment” discussed above in 3 which is interpreted in ($*$ -autonomous/closed) symmetric monoidal categories – a co-modality (def. 2.1) denoted “!” and called the *exponential modality*. The axioms on ! are roughly meant to be such that if A is a linear type, then $!A$ is the linear type obtained from it by universally equipping it with properties of a non-linear type. More precisely, by the general categorical semantics of (co-)modalities recalled above in def. 2.1, the exponential co-modality is to be interpreted as a some co-monad on the type system [BBHdP92], and the axioms on ! are such as to make its co-Kleisli category of co-free co-algebras be cartesian monoidal [See89] (see around prop. 17 of [Bi95] for more discussion).

Based on this in [Be95, Bi95] it is observed that generally the exponential modality is naturally interpreted as a comonad induced specifically from a strong monoidal adjunction between the given symmetric monoidal category $\text{Mod}(\ast)$ of linear types and some given cartesian closed monoidal category \mathbf{H} (representing a non-linear type system):

$$(L \dashv R) : \text{Mod}(\ast)^{\otimes} \begin{array}{c} \xleftarrow{L} \\ \xrightarrow{R} \end{array} \mathbf{H}^{\times} \quad ! := L \circ R.$$

(If only a strong monoidal functor L like this exists without necessarily a strong monoidal right adjoint, then [Bar97] speaks of the “structural fragment” of linear logic.)

Finally in [PoSh12] (4.3) it is observed that if $\text{Mod}(\ast)$ here is the fiber over the point of a linear type system Mod dependent over \mathbf{H} , as in 3.2, then, as our notation already suggests, there is a canonical choice for L induced from the dependent type structure, namely the map that sends $Y \in \mathbf{H}$ to the Y -dependent sum of the unit linear type $1_Y \in \text{Mod}(Y)$:

$$L : Y \mapsto \sum_Y 1_Y.$$

(On morphisms this L is given by the adjunction counit, we see this below in example 4.19 as a special case of the general secondary integral transform formula). More generally, the dependent linear homotopy-type theory induces for each $X \in \mathbf{H}$ a functor

$$L_X : \mathbf{H}_{/X} \longrightarrow \text{Mod}(X)$$

given by summing the unit linear type along the fibers of f :

$$L_X : (Y \xrightarrow{f} X) \mapsto \sum_f 1_Y.$$

In this dependent form one recognizes this as the linear version of the operation considered in section 2 (p. 12) of [Law70b]. There the condition that these functors have right adjoints R_X is found to be the categorical semantics of the foundational *axiom of comprehension* (*axiom of separation*). Therefore one may say that dependent linear type theory carries a canonical $!$ -modality precisely if it satisfies the linear version of the comprehension axiom. Since the comprehension axiom in foundations is typically taken for granted, this shows that the existence of the exponential modality is a rather fundamental phenomenon.

In [BPS94] it had been found that in familiar models of (multiplicative) linear type theory such as in the category of vector spaces, the exponential sends a vector space to its *Fock space*, the vector space underlying the free symmetric algebra on the given space. This construction is manifestly a categorified exponential. Now in the simplistic 1-categorical model given by vector spaces over sets, example 3.14, the left adjoint $L = \sum 1_{(-)}$ canonically induced as above from the dependent type structure sends a set to the vector space it spans, R sends a vector space to its underlying set of vectors, and $! = L \circ R$ hence sends a vector space not quite to its Fock space, but to the space freely spanned by all the original vectors. (Another adjunction for vector spaces that does make $!$ produce exactly the Fock space is discussed in [Vi07].)

More interestingly;

Example 4.1. In the genuinely homotopy-theoretic model of linear homotopy-type theory given by E -module spectra over ∞ -groupoids, example 3.11, then $L = E \wedge \Sigma_+^\infty$ sends a homotopy-type to its suspension spectrum, and $R = \Omega^\infty$ sends a spectrum to its underlying infinite-loop space.

There is a deep sense in which stable homotopy theory is analogous to linear algebra, namely Goodwillie's calculus of functors (see section 7 of [LurieAlg]), and it has been argued in [ArKa95] that from this point of view at least $\Omega^\infty \circ \Sigma_+^\infty$ is indeed the analogue of the exponential function.¹³

In conclusion we find that the exponential modality $!$ in linear type theory, when implemented in linear homotopy-type theory naturally decomposes into an adjunction whose left adjoint is the process of forming spaces of sections, hence quantum states (and whose induced comonad encodes free second quantization).

Below in 7 we find that the process of quantization of an action functional $\exp(\frac{i}{\hbar}S)$ subject to a consistent (anomaly free) choice of path integral measure $d\mu$ is given by a twisted variant of the canonical left adjoint L as above: we find there that a choice of action functional comes with a choice of an assignment of invertible linear homotopy-types $A_X \in \text{Mod}(X)$ (thought of as “dual prequantum line bundles”) to homotopy-types $X \in \mathbf{H}$, and the quantization process sends these to their space of dual sections:

$$X \mapsto A_X \mapsto \sum_X A_X \in \text{Mod}(*).$$

For later reference we note at this point that

Remark 4.2. If we think of A_X as a linear bundle over X (as in example 3.10 via example 2.26) then its X -dependent product is to be thought of as its linear space of sections

$$\Gamma_X(A) := \prod_X A.$$

If A here is dualizable with dual $\mathbf{L} := \mathbb{D}A$, then by dependent linear de Morgan duality, prop. 3.18 this is equivalently the linear dual

$$\Gamma_X(\mathbf{L}^*) \simeq \mathbb{D} \sum_X \mathbf{L}.$$

¹³ I am grateful to Mike Shulman and to David Corfield for highlighting this point and in fact for driving it home with some patience.

In the standard models of linear homotopy-type theory such as that of definition 3.11, the dependent sum $\sum_X \mathbf{L}$ has the interpretation of the *compactly supported* sections of \mathbf{L} (this is of course the default interpretation of \sum in the traditional yoga of six functors, see the citations in [May05]). Therefore the linear dual $\mathbb{D}\sum_X \mathbf{L}$ is interpreted as the space of distributional sections. In a general Verdier-Grothendieck context of six functors these may be different from the genuine sections (of the dual bundle), but here in the Wirthmüller context they coincide.

In the terminology of [Law86] the type of states $\prod_X \mathbf{L}^*$ would be “intensive”, while $\sum_X \mathbf{L}$ would be “extensive”.

We now turn to the definition of fundamental classes that allow to integrate such sections against a measure and then we use this to define secondary integral transforms acting on spaces of sections.

4.3 Fundamental classes and Measures

We discuss here how to axiomatize basic measure theory in dependent homotopy-type theory, such as to define integrals of sections of linear types (remark 4.2) along maps of contexts that are equipped with a measure. In terms of linear logic and linear type theory the construction here is a variant and generalization of the duality structure reflected by the orthocomplementarity in the original BvN quantum logic [BvN36] and more generally in “dagger-structure” in linear type theory. We postpone discussion of this to 4.5. Let throughout

$$\begin{array}{c} \text{Mod}(-) \\ \downarrow \\ \mathbf{H} \end{array}$$

be a model for linear homotopy-type theory, def. 3.6.

Definition 4.3. A *fiberwise twisted fundamental class* on a morphism $f : X \rightarrow Y$ in \mathbf{H} is (if it exists) a choice of dualizable object $\tau \in \text{Mod}(Y)$ (the twist) such that $f_! f^* \mathbb{D}\tau$ is dualizable, together with a choice of equivalence of the form

$$f_! f^*(1_Y) \xrightarrow{\simeq} \mathbb{D}(f_! f^*(\mathbb{D}\tau)) \simeq f_* f^*(\tau).$$

Remark 4.4. This is a specialization of the assumption in (4.3) of [May05]. There it is emphasized that for many constructions the assumption that $f_! f^* 1_X$ be dualizable is, while typically verified in applications, not necessary. However, it is necessary for the definition of secondary integral transforms in def. 4.18 below, and therefore we do require it.

Remark 4.5. If here $Y \simeq *$ is the terminal object, so that $f : X \rightarrow *$ is the essentially unique terminal morphism, then we usually denote a fundamental class on this morphism by $[X]$.

Proposition 4.6 (Wirthmüller isomorphism). *Given a fiberwise fundamental class, def. 4.3, on a morphism $f : X \rightarrow Y$, then for dualizable $A \in \text{Mod}(Y)$ there is a canonical natural equivalence*

$$f_* f^* \mathbb{D}A \simeq \mathbb{D}(f_! f^* A) \xrightarrow{\simeq} f_! f^*((\mathbb{D}A) \otimes (\mathbb{D}\tau))$$

and hence a canonical natural transformation

$$f_! f^* A \rightarrow \mathbb{D}(f_! f^*((\mathbb{D}A) \otimes (\mathbb{D}\tau))) \simeq f_* f^* \mathbb{D}((\mathbb{D}A) \otimes (\mathbb{D}\tau))$$

which is an equivalence if $f_! f^* A$ is dualizable.

Proof. The canonical map itself is the composite

$$\begin{array}{c}
f_* f^* \mathbb{D}A \\
\downarrow \simeq \\
(f_* f^* \mathbb{D}A) \otimes (\mathbb{D}1_Y) \longrightarrow (f_* f^* \mathbb{D}A) \otimes (\mathbb{D}f_! f^* 1_Y) \\
\downarrow \simeq \\
(f_* f^* \mathbb{D}A) \otimes (f_! f^* \mathbb{D}\tau) \\
\downarrow \simeq \\
f_!((f^* f_* f^* \mathbb{D}A) \otimes (f^* \mathbb{D}\tau)) \longrightarrow f_!((f^* \mathbb{D}A) \otimes (f^* \mathbb{D}\tau)) \\
\downarrow \simeq \\
f_! f^* (\mathbb{D}A \otimes \mathbb{D}\tau)
\end{array}$$

where the first is the unit equivalence for $1_Y \simeq \mathbb{D}1_Y$, the second is tensoring with the dual of the $(f_! \dashv f^*)$ -counit, the third is tensoring with the dual of the defining equivalence of a fundamental class, the fourth is the projection formula of Frobenius reciprocity, def. 3.2, the fifth comes from the $(f^* \dashv f_*)$ -counit and the last one finally is the strong monoidalness of f^* .

That this total composite is an equivalence is prop. 4.13 in [May05], specialised to twists of the form as in def. 4.3, following remark 4.4. \square

Remark 4.7. If the twist in prop. 4.6 vanishes (is the tensor unit) then a Wirthmüller isomorphism means that $f_!$ coincides with f_* on objects in the image of f^* . Since $f_!$ is the left adjoint and f_* the right adjoint of f^* , this means that in this situation f^* has a two-sided adjoint, hence an “ambidextrous” adjoint. The condition $f_! \simeq f_*$ together with a further coherence condition is called “ambidexterity” in Construction 4.1.8 of [HopkinsLurie14].

The central construction obtained from a Wirthmüller-type six-operations context that we need below is now the following.¹⁴ Let $f : X \rightarrow Y$ be a morphism of contexts in a linear homotopy-type theory with associated base change Wirthmüller morphism $(f_! \dashv f^* \dashv f_*) = (\sum_X \dashv f^* \dashv \prod_X)$, def. 3.1,

Definition 4.8. Given a fiberwise fundamental class, def. 4.3, on f and given $A \in \text{Mod}(Y)$ a dualizable object such that also $f_! f^* A$ is dualizable, write

$$[f] : (-) \otimes \tau \rightarrow f_! f^*(-)$$

for the natural transformation given as the composite

$$[f]_A : A \otimes \tau \xrightarrow{\mathbb{D}(\epsilon_{\mathbb{D}(A \otimes \tau)})} \mathbb{D}f_! f^*(\mathbb{D}(A \otimes \tau)) \xrightarrow{\simeq} f_! f^* A ,$$

where the first morphism is the dual of the $(f_! \dashv f^*)$ -counit on $\mathbb{D}(A \otimes \tau)$, and where the second morphism is the equivalence of prop. 4.6.

We will usually also refer to $[f]$ as the fundamental class, whence the notation.

The dual of the Y -dependent sum of the fundamental class we call the induced *measure* on f and write

$$d\mu_f(A) := \mathbb{D} \left(\sum_Y [f]_A \right) : \mathbb{D} \left(\sum_Y f_! f^* A \right) \rightarrow \mathbb{D} \left(\sum_Y A \otimes \tau \right) .$$

¹⁴While this text was being composed, essentially def. 4.8 for the special case of vanishing twist and in the specific model of ∞ -module bundles over homotopy types appeared as Construction 4.0.7 and Notation 4.1.6 of [HopkinsLurie14].

Remark 4.9. By remark 3.19 the fundamental class in def. 4.8 is equivalently the composite

$$[f]_A : A \otimes \tau \xrightarrow{\eta_{A \otimes \tau}} f_* f^*(A \otimes \tau) \xrightarrow{\cong} f_! f^* A .$$

In this form the fundamental class here is manifestly related to what appears in remark 4.1.7 of [HopkinsLurie14], we come back to this below in example 6.3.

Remark 4.10. The fundamental class morphism in def. 4.8 is reverse to the $(f_! \dashv f^*)$ -counit. We may think of it as the “Umkehr map” of the counit and find in the following that this naturally induces Umkehr maps for other morphisms.

Remark 4.11. A measure on a map $f : X \rightarrow Y$ in def. 4.8 is to be thought of as a Y -parameterized collection of measures on all of the (homotopy-)fibers of the map. This is explicitly so in the internal linear logic, in which the fundamental class reads

$$y : Y, A(y) : \text{Type} \vdash [f](y) : A(y) \otimes \tau(y) \rightarrow \sum_{x \in f^{-1}(y)} A(f(x)) .$$

Externally this fiberwise property is directly visible in the model of dependent homotopy type theory given by bundles of spectra over ∞ -groupoids, this is prop. 4.3.5 in [HopkinsLurie14].

4.4 Correspondences and Secondary integral transforms

We discuss now how a correspondence of contexts in linear homotopy-type theory which is equipped with a fiberwise fundamental class on its right leg and with a linear map between linear homotopy-types pulled back to its correspondence space (a secondary integral kernel) naturally induces a secondary integral transform.

Examples include ordinary matrices in linear algebra, example 6.1, pull-push in twisted generalized cohomology by twisted Pontryagin-Thom Umkehr maps, example 6.2 and in particular the “ambidexterity” in stable homotopy theory of [HopkinsLurie14], example 6.3.

Throughout, let Mod be a linear homotopy-type theory. A secondary linear integral transform is supposed to be a linear function between linear spaces of sections, remark 4.2, which is induced from an integral kernel or matrix given by a linear map between linear bundles L over some correspondence space.

Definition 4.12. Given a dependent linear homotopy-type theory Mod , then a *prequantum integral kernel* is a correspondence

$$\begin{array}{ccc} & Z & \\ i_1 \swarrow & & \searrow i_2 \\ X_1 & & X_2 \end{array}$$

of contexts – the *arity* – together with linear types $A_1 \in \text{Mod}(X_1)$ and $A_2 \in \text{Mod}(X_2)$ – the *coefficients* – and a linear function of the form

$$\xi : i_1^* A_1 \leftarrow i_2^* A_2 ,$$

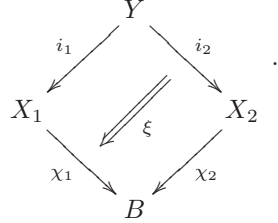
the *integral kernel* itself. A *quantum integral kernel* or *amplimorphism* is a correspondence and linear types as above and a morphism

$$\Xi : \sum_Z i_1^* A_1 \leftarrow \sum_Z i_2^* A_2 .$$

Remark 4.13. So if ξ is a pre-quantum integral kernel then $\Xi := \sum_Z \xi$ is the corresponding quantum integral kernel.

Further below in prop. 7.9 we find a more abstract, more conceptual origin of prequantum integral kernels. For the moment we are content with pointing out that a typical source of prequantum integral kernels are correspondences dependent on a context of moduli for certain linear types [Sch13a, Nuiten13]:

Example 4.14. Let B be some base context and $V \in \text{Mod}(V)$ a B -dependent linear type. Then every correspondence in $\mathbf{H}/_B$ canonically induces an invertible prequantum integral kernel, def. 4.12 as follows. In \mathbf{H} (under dependent sum) the correspondence in $\mathbf{H}/_B$ is a diagram of the form



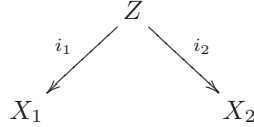
Hence putting

$$A_i := \chi_i^* V$$

gives the prequantum integral kernel

$$\xi : (i_1)^* A_1 \xleftarrow{\simeq} (i_2)^* A_2 .$$

Example 4.15. For



a correspondence in \mathbf{H} and $A_2 \in \text{Mod}(X_1)$ any linear type, then setting

$$A_1 := \sum_{i_1} (i_2)^* A_1$$

yields a non-invertible prequantum integral kernel, where

$$\xi = \eta_{(i_2)^* A_1} : (i_1)^* \sum_{i_1} (i_2)^* A_1 \longleftarrow (i_2)^* A_1$$

is the unit of the $(\sum_{i_2} \dashv (i_2)^*)$ -adjunction. This we may call the universal (non-invertible) pre-quantization of the original correspondence and the given A_1 .

Example 4.16. In the model of linear homotopy-type theory given by an E_∞ -ring E as in example 3.11

$$\begin{array}{c} E\text{Mod}(-) \\ \downarrow \\ \text{Grpd}_\infty \end{array}$$

a function between two linear types is a cocycle in bivariant generalized E -cohomology. Therefore in this case a prequantum integral kernel as in def. 4.12 is a correspondence equipped with a cocycle on its correspondence space. This is, broadly, the structure of *motives*. Indeed, we see below in 6.2 that the secondary integral transform in $E\text{Mod}$ for $E = \text{KU}$ may be given by KK-theory classes which were argued by Alain Connes to be the K-theoretic analog of motives, a point of view that has been made precise in [Mahanta13]. We discuss this analogy a bite more below in 8.2.

In order to apply an integral kernel as a linear map to the spaces of sections of its coefficient bundles, the idea is to “pull” these sections up along one of the two legs of the correspondence, apply there the map that defines the integral kernel, and then “push” the result down along the other leg. Notice that:

Remark 4.17. For $\mathbb{S} \in \mathbf{CRing}_\infty$ the sphere spectrum regarded as an E_∞ -ring let $\mathbf{SMod}(-)$ be the corresponding model of linear homotopy-type theory from example 3.11. Given $f : X \rightarrow Y$ a morphism in $\infty\mathbf{Grpd}$ then forming the suspension spectra yields a morphism of the form

$$\Sigma_+^\infty f : \Sigma_+^\infty X \rightarrow \Sigma_+^\infty Y.$$

As in example 4.1 we have that

$$\Sigma_+^\infty X \simeq \sum_X 1_X,$$

where now 1_X is the trivial spherical fibration (trivial \mathbb{S} -line bundle) over X . Under this identification the above morphism is given by the $(\sum_X \dashv X^*)$ -counit ϵ :

$$\Sigma_+^\infty X \simeq \sum_X 1_X \simeq \sum_Y \sum_f f^* 1_Y \xrightarrow{\sum_Y \epsilon_{1_Y}} \sum_Y 1_Y \simeq \Sigma_+^\infty Y.$$

Generally for $E \in \mathbf{CRing}_\infty$ any E_∞ -ring, then this construction yields the map that is called “pushforward in generalized E -homology” along f

$$E_\bullet(X) \rightarrow E_\bullet(Y).$$

The image of this under dualization \mathbb{D} is the “pullback in generalized E -cohomology” along f

$$E^\bullet(Y) \rightarrow E^\bullet(X).$$

Beware that these operations are often denoted by “ f_* ” and “ f^* ”, respectively, but that for us these symbols denote push/pull not of sections but of the E -module bundles themselves, and that sections are pull/pushed instead via the (dual of) the counit, as above.

Our central notion is now the following, which generalizes the above to general linear homotopy-type theory, general twists and combines it with Umkehr maps in order to produce a secondary “pull-tensor-push integral transform” on cohomology.¹⁵

Definition 4.18. Given a prequantum integral kernel ξ or quantum kernel Ξ as in def. 4.12 and a fiberwise fundamental class, def. 4.3, on the right leg i_2 , with induced fundamental class $[i_2]$, def. 4.8, then we say that the morphism

$$\mathbb{D} \int_Z \Xi d\mu_{i_2} : \sum_{X_1} A_1 \xleftarrow{\sum_{X_1} \epsilon_{A_1}} \sum_{X_1} (i_1)_! (i_1)^* A_1 \xleftarrow{\simeq} \sum_Z (i_1)^* A_1 \xleftarrow{\Xi} \sum_Z (i_2)^* A_2 \xleftarrow{\simeq} \sum_{X_2} (i_2)_! (i_2)^* A_2 \xleftarrow{\sum_{X_2} [i_2]_{A_2}} \sum_{X_2} A_2 \otimes \tau$$

is the induced *dual secondary integral transform*. The dual morphism

$$\int_Z \Xi d\mu_{i_2} : \mathbb{D} \sum_{X_1} A_1 \rightarrow \mathbb{D} \sum_{X_2} (A_2 \otimes \tau)$$

we call the corresponding *secondary integral transform*.

¹⁵While this text was being composed, essentially def. 4.18 for the special case of vanishing twist and in the specific model of ∞ -module bundles over homotopy types appeared as Notation 4.1.6 of [HopkinsLurie14].

Example 4.19. Consider the simple case of a prequantum integral kernel whose underlying correspondence has as right leg an identity

$$\begin{array}{ccc} & X & \\ d \swarrow & & \searrow \text{id} \\ Y & & X \end{array},$$

where the linear types on the base spaces are the unit types and the integral kernel itself is the identity $\xi = \text{id} : f^*1_Y = 1_X \rightarrow 1_X = \text{id}^*1_X$. Then the right leg id_X is trivially oriented with vanishing twist and with this choice the secondary integral transform formula in def. 4.18 reduces to being map

$$\mathbb{D} \int_X d\mu_{\text{id}} : \sum_Y 1_Y \xleftarrow{\sum_Y^\epsilon} \sum_Y f! f^* 1_X \xleftarrow{\sim} \sum_X 1_X.$$

This we recognize as the operation considered in (4.3) of [PoSh12]. We had discussed the meaning of this operation in 4.2 above.

Remark 4.20. If the coefficients A_1 and A_2 in def. 4.18 are dualizable with duals L_1 and L_2 , respectively, then by linear de Morgan duality, prop. 3.18, and by remark 4.2, the secondary integral transform of def. 4.18 is a linear function between the linear spaces of sections of the dual coefficients:

$$\int_Z \Xi d\mu_{i_2} : \Gamma_X(L_1) \longrightarrow \Gamma_Y(L_2 \otimes \mathbb{D}\tau).$$

Example 6.1 below shows how basic linear algebra is a special case of def. 4.18. This is elementary in itself, but turns out to be directly the blueprint for the more sophisticated example 6.2 to follow, which in turn is the context in which one finds genuine quantum physics by example 7.3. In view of these examples, we make the following observation on the conceptual interpretation of the construction in def. 4.18, which the reader with no tolerance for more philosophical considerations is urged to skip and ignore.

Remark 4.21 (logical interpretation of the secondary integral transform). These examples show that we may think of X and Y in def. 4.18 as phase spaces and, if A_1 and A_2 are dualizable, think of the linear types L_1 and L_2 as pre-quantum line bundles on these (see also section 1.2.10 in [Sch13a], surveyed in [Sch13b]). Hence by the BHK correspondence (as reviewed in 2.1), in the underlying linear logic L_1 represents a proposition about elements of X (and L_2 about elements of Y): L_1 may be thought of as the linear proposition that the given system is in state $x \in X$ of its phase space. For a proposition in classical logic the fiber of L_1 over some $x \in X$ would be either empty or inhabited, indicating that the system either is in that state or not. Now in linear logic this fiber is a linear space, namely what in physics is a space of *phases*. In this vein we have the following stages of interpreting the expression $\sum_X L_1$:

1. in logic this expression is the existential quantification $\exists_{x \in X} L_1(x)$ asserting that “there is a state x occupied by the physical system”;
2. in type theory this expression denotes the collection (type) of all states that the system can be in;
3. in homotopy-type theory this expression denotes the homotopy-type of all such states, hence properly taking their gauge equivalences and higher gauge equivalences into account;
4. finally in linear homotopy-type theory this expression is the *linear* space of all states (with gauge equivalence taken into account) obtained not by disjointly collecting them all but by linearly adding up their phases.

An analogous comment applies to the middle terms in the composite function in 4.18, $\Xi = \sum_Z \xi$. Here now the correspondence space Z is to be interpreted as a space of paths (trajectories) from X and Y , with $z \in Z$ being a path going from $p_1(z) \in X$ to $p_2(z) \in Y$. Hence in analogy to the above we have that $\sum_Z \xi$ has the following interpretations:

1. in logic it means “that there is a path”;
2. in type theory it means “the collection of all paths”;
3. in homotopy-type theory it means “the collection of all paths with gauge transformations accounted for”;
4. finally in linear homotopy-type theory it means “the sum of the phases of all possible paths”.

4.5 Dagger-structure, Fiberwise inner products and Quantum operations

Above in 3.1 we discussed how quantum logic is linear logic, the logic of closed symmetric monoidal categories. For core constructions in quantum physics and quantum computation, one considers an additional structure on these categories, namely what is called a *strongly compact* [AbCo04] or *dagger-compact* (\dagger -compact) structure [Sel07]. Here we discuss how the concept of fundamental classes in dependent linear type theory that we introduced in 4.3 naturally induces \dagger -structure in the special case where the twist vanishes. Conversely, we may hence regard the concept of fundamental classes in def. 4.3 as a generalization of \dagger -structure.

Remark 4.22. In the special case that the twist τ in a fiberwise fundamental class on f , def. 4.3 vanishes, in that $\tau \simeq 1_{\mathcal{D}}$, then this is then equivalent to an identification of the linear type

$$V_f := f_! f^*(1_{\mathcal{D}})$$

with its dual

$$V_f \xrightarrow{\simeq} V_f^*.$$

This way an untwisted fiberwise fundamental class on f is equivalently a non-degenerate inner product

$$\langle -, - \rangle : V_f \otimes V_f \longrightarrow 1_{\mathcal{D}}.$$

In this spirit we say that:

Definition 4.23. For $A \in \text{Mod}(X)$ dualizable, a choice of *fiberwise inner product* is a choice of equivalence

$$A \xrightarrow{\simeq} \mathbb{D}A.$$

If this is the inverse of its dual morphism, we say the inner product is *symmetric* (axiom (T5) in [Sel10]).

The corresponding pairing we write

$$\langle -, - \rangle_A : A \otimes A \xrightarrow{\simeq} A \otimes \mathbb{D}A \xrightarrow{\text{ev}} 1_X$$

and often we find it convenient to use “ $\langle -, - \rangle_A$ ” also for the original equivalence itself. In this notation the symmetry condition is that $\langle -, - \rangle_A \simeq \mathbb{D}\langle -, - \rangle_A^{-1}$.

If $X \simeq *$ we may call a fiberwise inner product over X just an “inner product” or “global inner product”, for emphasis. The following examples show how a fiberwise inner product induces a global one.

Example 4.24. If $A \in \text{Mod}(X)$ is equipped with a fiberwise inner product $\langle -, - \rangle_A$, def. 4.23, and if X (hence the terminal morphism $X \rightarrow *$) is equipped with an untwisted fundamental class $[X]$, def. 4.3, then $\sum_X A \in \text{Mod}(*)$ is naturally equipped with the inner product given by the composite

$$\langle -, - \rangle_{\sum_X A} : \sum_X A \xrightarrow{\sum_X \langle -, - \rangle_A} \sum_X \mathbb{D}A \xrightarrow{\simeq} \prod_X \mathbb{D}A \xrightarrow{\simeq} \mathbb{D} \sum_X A,$$

where the second equivalence is the Wirthmüller isomorphism induced by the fundamental class (by the second clause in prop. 4.13 of [May05], using that $\sum_X 1_X$ is dualizable by our assumption on fundamental classes, see remark 4.4) and the last one is parameterized linear De Morgan duality, prop. 3.18.

Example 4.25. If $A \in \text{Mod}(X)$ is equipped with a fiberwise inner product, def. 4.23, and $f : Y \rightarrow X$ is equipped with an untwisted fiberwise fundamental class, def. 4.3, then this induces on $f_! f^* A$ a fiberwise inner product given as the composite

$$\langle -, - \rangle_{f_! f^* A} : f_! f^* A \xrightarrow{\simeq} \mathbb{D} f_! f^* \mathbb{D} A \xrightarrow[\simeq]{\mathbb{D} f_! f^* \langle -, - \rangle_A^{-1}} \mathbb{D} f_! f^* A$$

of the induced Wirthmüller isomorphism, prop. 4.6, and the image of the fiberwise fundamental class under $\mathbb{D} f_! f^*(-)$.

A simple but fundamental fact is that between objects that are equipped with (fiberwise) inner products, every morphism has a canonical reversal:

Definition 4.26. Given a morphism $f : A \rightarrow B$ between linear types equipped with fiberwise inner product, def. 4.23, then we say its *transpose* f^\dagger is the composite

$$f^\dagger : B \xrightarrow[\simeq]{\langle -, - \rangle_B} \mathbb{D} B \xrightarrow{\mathbb{D} f} \mathbb{D} A \xrightarrow[\simeq]{\langle -, - \rangle_A^{-1}} A .$$

Some comments on this basic abstract construction of \dagger -structure are in section 4 of [Sel10].

We may now relate the choice of a fiberwise fundamental class to the transpose of the pushforward along the map.

Proposition 4.27. *Let $A \in \text{Mod}(X)$ be dualizable and equipped with a fiberwise symmetric inner product $\langle -, - \rangle_A$, def. 4.23, and let $f : Y \rightarrow X$ be a morphism of contexts equipped with an untwisted fiberwise fundamental class, def. 4.3. Then the respective morphism $[f]$, def. 4.8, is the transpose, def. 4.26, of the $(\sum_f \dashv f^*)$ -counit:*

$$[f] \simeq \epsilon_f^\dagger ,$$

hence, by remark 4.9

$$\epsilon_f^\dagger \simeq \eta_f .$$

Proof. By naturality of the counit we have

$$\begin{array}{ccc} & & [f]_A \\ & \curvearrowright & \\ A & \xrightarrow{\mathbb{D} \epsilon_{\mathbb{D} A}} & \mathbb{D} f_! f^* \mathbb{D} A \xrightarrow{\simeq} f_! f^* A , \\ \mathbb{D} \langle -, - \rangle_A^{-1} \downarrow & & \downarrow \mathbb{D} f_! f^* \langle -, - \rangle_A^{-1} \nearrow \langle -, - \rangle_{f_! f^* A}^{-1} \\ \mathbb{D} A & \xrightarrow{\mathbb{D} \epsilon_A} & \mathbb{D} f_! f^* A \end{array}$$

where the square on the left is the image under \mathbb{D} of the naturality square of the $(f_! \dashv f^*)$ -counit on the fiberwise inner product $\langle -, - \rangle_A^{-1} : \mathbb{D} A \xrightarrow{\simeq} A$, and where the diagonal equivalence on the right is the inverse of the map in example 4.25. By symmetry of the fiberwise inner product on X the left vertical map is equivalent to $\langle -, - \rangle_A$ and hence the bottom composite of the diagram exhibits $[f]_A$ as the transpose of $\mathbb{D} \epsilon_A$. \square

Corollary 4.28. *If X itself (hence $X \rightarrow *$) is equipped with an untwisted fundamental class $[X]$ then*

$$\sum_X [f] \simeq \left(\sum_X \epsilon \right)^\dagger$$

Proof. Combining example 4.24 and prop. 4.27. □

Therefore:

Remark 4.29. If $A \in \text{Mod}(X)$ is equipped with a fiberwise symmetric inner product $\langle -, - \rangle_A$ and $f : Y \rightarrow X$ is equipped with untwisted fiberwise fundamental classes, def. 4.3, then the formula for the secondary integral transform $\mathbb{D} \int_Z \Xi d\mu$ in def. 4.18 of a prequantum integral kernel on a correspondence

$$\begin{array}{ccc} & Z & \\ i_1 \swarrow & & \searrow i_2 \\ X_1 & & X_2 \end{array}$$

becomes

$$\mathbb{D} \int_Z \Xi d\mu \simeq \sum_X \epsilon_{A_1}^{i_1} \circ \Xi \circ \sum_X (\epsilon_{A_2}^{i_2})^\dagger.$$

If moreover X itself is equipped with a fundamental class then this becomes

$$\mathbb{D} \int_Z \Xi d\mu \simeq \left(\sum_X \epsilon_{A_1}^{i_1} \right) \circ \Xi \circ \left(\sum_X \epsilon_{A_2}^{i_2} \right)^\dagger.$$

This kind of operation plays a special role both in abstract quantum physics as well as in generalized cohomology theory:

Remark 4.30. In particular for the case that $i_1 = i_2$ and $A_1 = A_2$ (so that in example 6.1 the integral kernel is a square matrix) then the map

$$\Xi \mapsto \mathbb{D} \int_Z \Xi d\mu \simeq \left(\sum_X \epsilon_A \right) \circ \Xi \circ \left(\sum_X \epsilon_A \right)^\dagger$$

(which we identify as the path integral quantization map for the integral kernel Ξ) is what is called a (completely positive) “quantum operation”, see [Sel07].

Remark 4.31. In the model of linear homotopy-type theory by generalized cohomology theory, def. 3.11, the self-duality of 4.23 is *Poincaré duality* (in general with a twist) and the induced transpose maps in def. 4.26 are the “Umkehr maps” or “wrong way maps” in generalized cohomology.

Specifically the literature on KK-theory knows that forming Umkehr maps in K-theory is given by forming transpose morphisms of the “right way”-morphisms in the symmetric monoidal category KK, see [BMRS07]. Definition 2.1 in [BMRS07] defines (somewhat implicitly) a fundamental class to be a choice of self-duality in KK (Poincaré duality in KK) and section 3.3 there defines construction of Umkehr maps as the corresponding construction of transposes, hence of the dagger-operation as in def. 4.26. Under this identification the reformulation of secondary integral transforms via dagger operations in remark 4.29 corresponds to formula (5.6) in [BMRS07].

5 Directed homotopy-type theory

We have discussed in 3.1 how quantum mechanics is faithfully axiomatized in linear (homotopy-)type theory; and have indicated that this is not sufficient for the description of modern quantum physics, as it misses aspects of quantum field theory. One may think of quantum mechanics as being 1-dimensional quantum field theory, and conversely of local quantum field theory as being a refinement from quantum propagation along one dimension (time) to a description of propagation locally in more directions (space and time) (see

[SaSc11] for review and pointers to the literature). This higher dimensional directionality is formalized, semantically, in the theory of monoidal n -categories (meaning: (∞, n) -categories). For $n = 1$ these coincide with the categories (∞ -categories) that we saw in 2.1 provide semantics for homotopy-type theory. Therefore the generalization of this to (∞, n) -categories has been called *directed homotopy-type theory*.

In this note we do not dwell too much on this except to make here a remark on what the homotopy-theoretic structure of local quantum field theory is in a little more detail, just enough so that it becomes clear how the linear homotopy-type theory which we considered so far sits inside there, and hence how it is that we may connect to the process of quantization, which is the topic of 7 below.

5.1 Free directed linear homotopy-types and Quantum field theory

In the traditional Schrödinger picture, a d -dimensional quantum field theory Z is given by assigning to each compact space Σ_{d-1} a complex vector space (Hilbert space) $Z(\Sigma_{d-1})$ (of quantum states of fields on Σ_{d-1}) and to each cobordism $\Sigma_d : \Sigma_{d-1}^{\text{in}} \rightarrow \Sigma_{d-1}^{\text{out}}$ (“spacetime” or “worldvolume”) a linear map

$$Z(\Sigma_d) : Z(\Sigma_{d-1}^{\text{in}}) \longrightarrow Z(\Sigma_{d-1}^{\text{out}})$$

(propagating quantum states along Σ) such that this assignment is functorial and such that it is monoidal, in that the disjoint union of spaces is sent to the tensor product of vector spaces. Hence such a non-localized topological quantum field theory is a symmetric monoidal functor of the form

$$Z : \text{Bord}_n^{\text{II}} \longrightarrow \text{CMod}^{\otimes}.$$

The locality of such a quantum field theory is encoded by refining this to an (∞, n) -functor

$$Z : \text{Bord}_n^{\text{II}} \longrightarrow E\text{Mod}_n^{\otimes},$$

where $E \in \text{CRing}_{\infty}$ is a suitable E_{∞} -ring and $E\text{Mod}_n$ is an (∞, n) -category of n -fold E -modules, hence of higher “directed” types of quantum states, this we come to in 5.2 and 5.3 below.

The central theorem of [Lurie09a] says that the n -category Bord_n is the free symmetric monoidal (∞, n) -category on the collection of boundary data (branes) and defect data (domain walls). Hence linear-type theoretically one may think of Bord_n as a collection of free types in multiplicative linear directed homotopy type theory.

5.2 Directed types of quantum states

Consider a model

$$\begin{array}{c} \text{Mod} \\ \downarrow \\ \mathbf{H} \end{array}$$

for linear homotopy-type theory, def. 3.6. We observe that this naturally comes equipped with a higher directed notion of linear types, too.

Definition 5.1. For every type $X \in \mathbf{H}$ the symmetric monoidal category $\text{Mod}(X)$ is canonically a module category over the symmetric monoidal category $\text{Mod}(\ast)$, via the action

$$\text{Mod}(\ast) \times \text{Mod}(X) \longrightarrow \text{Mod}(X)$$

given by

$$(\tau, A) \mapsto (X^* \tau) \otimes A.$$

A functor $F : \text{Mod}(X) \longrightarrow \text{Mod}(Y)$ is called $\text{Mod}(\ast)$ -linear if it respects this action.

Remark 5.2. That def. 5.1 indeed defines an action is equivalent to the fact that

$$X^* : \text{Mod}(\ast) \longrightarrow \text{Mod}(X)$$

is a strong monoidal functor, by the axioms of linear homotopy-type theory, def. 3.6.

Proposition 5.3. For $f : X \longrightarrow Y$ any map in \mathbf{H} , then pullback

$$f^* : \text{Mod}(X) \longleftarrow \text{Mod}(Y)$$

is a $\text{Mod}(\ast)$ -linear functor, def. 5.1, as is the sum along the fibers of f

$$\sum_f : \text{Mod}(X) \longrightarrow \text{Mod}(Y).$$

Proof. For $A \in \text{Mod}(X)$ and $\tau \in \text{Mod}(\ast)$ we naturally have

$$\begin{aligned} f^*(\tau \cdot A) &= f^*((Y^*\tau) \otimes A) \\ &\simeq (f^*Y^*\tau) \otimes f^*A \\ &\simeq (Y^*\tau) \otimes f^*A \\ &= \tau \cdot f^*A \end{aligned}$$

where we used that f^* is strong monoidal, and

$$\begin{aligned} \sum_f(\tau \cdot A) &= \sum_f((X^*\tau) \otimes A) \\ &\simeq \sum_f((f^*Y^*\tau) \otimes A) \\ &\simeq (Y^*\tau) \otimes \sum_f A \\ &= \tau \cdot \sum_f A. \end{aligned}$$

where the last equivalence is Frobenius reciprocity. □

To reflect this we may say:

Definition 5.4. Write

$$\text{Mod}_2 \in (\infty, 2)\text{Cat}$$

for the $(\infty, 2)$ -category of $\text{Mod}(\ast)$ -linear ∞ -categories of the form $\text{Mod}(X)$ for some $X \in \mathbf{H}$, and $\text{Mod}(\ast)$ -linear functors between them.

In 5.3 below we expand a bit more on this (well-known) concept of higher categorical modules. In 7.3 we consider a kind of quantum field theory that does have directed spaces of quantum states given by 2-modules of the form $\text{Mod}(X)$. For this to satisfy the axioms of a TQFT, we will need to require two extra properties on the ambient model for linear homotopy-type theory.

Definition 5.5. Given a model $\text{Mod}(-) \rightarrow \mathbf{H}$ for linear homotopy-type theory, def. 3.6, one says that it satisfies the *Beck-Chevalley condition* if for all ∞ -pullback squares in \mathbf{H}

$$\begin{array}{ccc} & Z & \\ h \swarrow & & \searrow f \\ X_1 & & X_2 \\ k \searrow & & \swarrow g \\ & Y & \end{array}$$

the composition

$$f_! h^* \longrightarrow f_! h^* k^* k_! \xrightarrow{\simeq} f_! f^* g^* k_! \longrightarrow g^* k_!$$

is an equivalence (between pull-push $\text{Mod}(X_1) \rightarrow \text{Mod}(X_2)$ along the upper half and push-pull along the lower half).

Example 5.6. The models for linear homotopy-type theory $\mathbf{H}^{\Delta^1} \xrightarrow{\text{cod}} \mathbf{H}$, example 3.8, and $E\text{Mod}(-) \rightarrow \infty\text{Grpd}$, example 3.11, satisfy the Beck-Chevalley condition, def. 5.5.

Proof. The first statement is equivalently the pasting law for ∞ -pullbacks in \mathbf{H} . The second appears as prop. 4.3.3 in [HopkinsLurie14]. \square

Definition 5.7. We say a model $\text{Mod}(-) \rightarrow \mathbf{H}$ for linear homotopy-type theory, def. 3.6, is *2-monoidal* if for all $X, Y \in \mathbf{H}$ we have

$$\text{Mod}(X \times Y) \simeq \text{Mod}(X) \otimes_{\text{Mod}(*)} \text{Mod}(Y)$$

Example 5.8. For $E \in \text{CRing}_\infty$ any E_∞ -ring, then the model of linear homotopy-type theory $E\text{Mod}(-) \rightarrow \infty\text{Grpd}$ is 2-monoidal, def. 5.7.

Proof. Since $X \in \infty\text{Grpd} \hookrightarrow (\infty, 1)\text{Cat}$ is small and $E\text{Mod}(* \in (\infty, 1)\text{Cat}$ is locally presentable, this follows from basic properties of the symmetric monoidal ∞ -category of locally presentable ∞ -categories [LurieAlg].¹⁶ \square

Proposition 5.9. In a model $\text{Mod}(-) \rightarrow \mathbf{H}$ for linear homotopy-type theory, def. 3.6, consider $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ in \mathbf{H} and $A_i \in \text{Mod}(X_i)$ and $B_i \in \text{Mod}(Y_i)$. Then

$$(f_1 \times f_2)^*(p_1^* B_1) \otimes (p_2^* B_2) \simeq (p_1^* f_1^* B_1) \otimes (p_2^* f_2^* B_2).$$

If the Beck-Chevalley condition, def. 5.5, holds then also

$$\sum_{f_1 \times f_2} (p_1^* A_1) \otimes (p_2^* A_2) \simeq (p_1^* \sum_{f_1} A_1) \otimes (p_2^* \sum_{f_2} A_2).$$

Proof. The first one follows immediately from the fact that pullback is required to be strong monoidal. The second one follows using Frobenius reciprocity and the Beck-Chevalley, as is shown in lemma 3.2 of [PoSh12]. \square

Corollary 5.10. If the given model for linear homotopy-type theory, def. 3.8, satisfies the Beck-Chevalley condition, def. 5.5 and is 2-monoidal, def. 5.7, then the $(\infty, 2)$ -functor

$$\text{TQFT}_{d+1} : \text{Corr}_1(\mathbf{H}) \longrightarrow \text{Mod}_2$$

given by sending correspondence to their linear polynomial functors, def. 3.20, is monoidal.

¹⁶ Thanks to Thomas Nikolaus for discussion of this point.

5.3 Tannaka duality for directed types of quantum states

We briefly recall here a definition of $(E\text{Mod}_n, \otimes)$, the n -fold directed higher refinement of categories of modules serving as higher directed analogs of spaces of quantum states, and explain the meaning of this definition via Tannaka duality.

For A a ring, consider the category $A\text{Mod}$ of its (left) modules. Tannaka duality (see for instance [Ve12] for review and pointers to the literature) describes the relation between properties of categories such as $A\text{Mod}$ and the underlying ring A . For instance if A is a commutative algebra, then $A\text{Mod}$ naturally inherits the structure of a monoidal category from the natural tensor product of A -modules. As such then $A\text{Mod}$ is itself a higher “directed” analog of a (semi-)ring. Therefore one may ask for suitable categories which are equipped with an action, in the suitable sense, of $(A\text{Mod}, \otimes_A)$. For instance if E is a commutative ring and A is equipped with the structure of an E -algebra, then $A\text{Mod}$ is naturally a module over the monoidal category $(E\text{Mod}, \otimes_E)$, with the action given by tensoring an E -module with an A -module over E . Such a module category is hence a higher “directed” analog of an ordinary module over a ring. We are going to take n -directed A -modules to be the result of iterating this reasoning n times. But in order to determine what “suitable” means in the above and how to encode these structures in a useful way, we invoke more Tannaka duality.

Being commutative is a very strong condition on a ring, and so one may ask what the necessary and sufficient structure on A is that makes its category $A\text{Mod}$ of modules be monoidal. The traditional textbook answer is that this is the structure of a *bialgebra* on A – however we need to be careful with the fine print of the Tannaka duality theorem for bialgebras: the theorem asserts an equivalence between bialgebras A over E and monoidal category $A\text{Mod}$ that are *equipped* with their forgetful functor (“fiber functor”) $A\text{Mod} \rightarrow E\text{Mod}$. Possibly more famous is the further refinement of this theorem to those bialgebras which are Hopf algebras, where Tannaka duality says that Hopf algebra structures on A over E are equivalent to autonomous monoidal category structure plus fiber functor on $A\text{Mod}$ (here “autonomous” means all objects have duals).

Only more recently did the full Tannaka duality for monoidal categories appear in the literature (see [Ve12] for a review), where the extra structure of a fiber functor is not assumed: say that a *sesquialgebra* [TWZ05] is an associative algebra A equipped with one more structure of a (co-)algebra, but not in the category $E\text{Mod}$ (which would make it an ordinary bialgebra), but in the more flexible 2-category $E\text{Mod}^b$ whose objects are associative E -algebras, but whose morphisms are bimodules between these (and whose 2-morphisms are bimodule homomorphisms). With this definition the full duality statement is obtained: the structure of a sesquialgebra on the associative E -algebra A is equivalent to the structure of a monoidal category on $A\text{Mod}$.

Observe that the 2-category of algebras with bimodules between then appearing also has another interpretation: the Eilenberg-Watts theorem says that this is equivalently the 2-category whose objects are categories of modules of the form $A\text{Mod}$, and whose morphisms are right exact functors between these that preserve colimits. Since colimits are the category-theoretic analog of addition, these may be thought of as the *linear* functors between the categories $A\text{Mod}$ regarded as $E\text{Mod}$ -modules. For this reason we may think of $E\text{Mod}^b$ as being $E\text{Mod}_2$, the 2-category of *directed E -modules* or *2-modules* over E ([Sch08], appendix). By the above discussion in 5.1, 2-modules in this sense should appear as higher order spaces of quantum states in 2-dimensional quantum field theory. This is indeed the case, see [Sch08] for a conceptual discussion in the sense used here and for further pointers to the literature.

Therefore if A is a sesquialgebra, then, by the above, we may say that it is an algebra object not just in the category $E\text{Mod}$ of E -modules, but in fact in the 2-category $E\text{Mod}_2$ of 2-modules over E . Since the sesquialgebra structure on A makes $A\text{Mod}$ be a monoidal category, we may apply Tannaka duality on this higher categorical level and regard $(A\text{Mod}, \otimes)$ as a stand-in for *its* 2-category of module categories. When this is regarded as having as morphisms bimodule-categories, as 2-morphisms module homomorphisms between these and as 3-morphisms natural transformation of these, then this forms a 3-category $E\text{Mod}_2^b$ which along the above lines we may regard as $E\text{Mod}_3$, the 3-category of 3-directed modules over E . For example this means that the Hopf algebras (quantum groups) that famously encode 3-dimensional Dijkgraaf-Witten

and Chern-Simons field theory (see for instance the review in [Ro05]), being in particular bialgebras and hence in particular sesquialgebras, are naturally stand-ins for 3-modules of quantum states, just as befits a 3-dimensional quantum field theory.

This process we may now iterate: we say that an n -module over E is a an $(n - 1)$ -fold algebra over E , which in turn means that it is an algebra object in the category of bimodules over algebra objects in the category of bimodules over ... over E . In this full generality the definition of $E\text{Mod}_n$ is indicated in [FHLT09]. A special strict version of the next step after sesquialgebra (bialgebra) and 3-modules, namely *trialgebras* Tannaka dual to certain 4-modules, has been considered in [Pf04] following [CrFr94].

We will not dwell on the higher module structures in this note here. For the purposes of the following section 7, the upshot of the above discussion is this: for $E \in \text{CRing}_\infty$ a commutative ring (E_∞ -ring), and for all $n \in \mathbb{N}$, then an n -module $N \in E\text{Mod}_n$ over E is in particular an ordinary E -module, equipped in addition with $(n - 1)$ compatible algebra structures, and morphisms between these include in particular n -module homomorphisms respecting the extra structure (while more generally the morphisms are bimodules, respecting extra structure).

5.4 Boundaries/branes and defects/domain walls

We consider in the following the first three stages, $d = 0, 1, 2$ of $(n + 1)$ -dimensional extended QFT. Moreover, we consider defects of codimension 1 and boundaries of codimension 2. Therefore, by the cobordism theorem for cobordisms with singularities, the data in dimension $0 \leq d \leq 2$ in principle determines everything else by higher traces. But we use coefficients only as suitable for a 2-dimensional QFT, namely the $(\infty, 2)$ -category of 2-modules from def. 5.4

$$Z : (\text{Bord}_n)_{0,1,2} \longrightarrow (\text{Corr}_n(\mathbf{H}))_{0,1,2} \longrightarrow \text{Mod}_2$$

By the above discussion this is necessary but of course not fully sufficient for constructing the fully extended $(n + 1)$ -dimensional QFT. In order to do so we would furthermore need to equip the object $Z(*)$ with the structure of an n -fold algebra, etc.

The basic singularity datum that we consider here is an elementary *corner*

$$\begin{array}{ccc} \text{Pic} & \xlongequal{\quad} & \text{Pic} \\ \uparrow & \swarrow \xi & \uparrow \mathbf{L} \\ * & \xleftarrow{\text{Fields}^\partial} & \text{Fields} \\ \parallel & & \downarrow \\ * & \xlongequal{\quad} & * \end{array} \quad .$$

Here **Fields** are the bulk fields of a d -dimensional field theory which we will denote by TQFT_d^τ , and which, by this diagram, appears as a defect that interpolates between the theory constant on Pic and the entirely trivial field theory in dimension $d + 1$. This defect itself has a boundary on the left, giving rise to a $(d - 1)$ -dimensional boundary field theory QFT_d with fields **Fields** $^\partial$

Then we also consider what it means to continue this further away from the boundary.

$$\begin{array}{ccccccc} \text{Pic} & \xlongequal{\quad} & \text{Pic} & \xlongequal{\quad} & \text{Pic} & \cdots & \\ \uparrow & \swarrow \xi & \uparrow \mathbf{L}_{\text{out}} & \swarrow \exp(\frac{i}{\hbar} S) & \uparrow \mathbf{L}_{\text{in}} & & \\ \widehat{\text{Pic}} & & * & \xleftarrow{\text{Field}^\partial} & \text{Fields}_{\text{out}} & \xleftarrow{(-)|_{\text{out}}} & \text{Fields}_{\text{traj}} & \xrightarrow{(-)|_{\text{in}}} & \text{Fields}_{\text{in}} & \cdots \\ \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ * & \xlongequal{\quad} & * & \xlongequal{\quad} & * & \cdots & * & \cdots & * & \cdots \end{array}$$

Composition to the right here imposes strong “anomaly cancellation” constraints on the quantization to make the composites functorial. But at the boundary itself, where no further composition is possible (to

the left) these conditions are relaxed and the available choices are part of what makes the boundary theory itself “geometric” (non-topological).

By the above, our truncated coefficients are such that these diagrams in \mathbf{H} are sent first to diagrams in Mod_2 of the form

$$\begin{array}{ccc}
\text{Mod}(\text{Pic}) & \xlongequal{\quad} & \text{Mod}(\text{Pic}) \\
\widehat{\text{Pic}}^* \downarrow & \swarrow & \downarrow \mathbf{L}_{\text{out}}^* \\
\text{Mod}(\ast) & \xrightarrow{\quad} & \text{Mod}(\mathbf{Fields}) \\
\parallel & \swarrow & \downarrow \sum_{\mathbf{Fields}} \\
\text{Mod}(\ast) & \xlongequal{\quad} & \text{Mod}(\ast)
\end{array}$$

For every choice of element in $\text{Mod}(\text{Pic})$, hence for every morphism $\text{Mod}(\ast) \rightarrow \text{Mod}(\text{Pic})$ this yields a diagram

$$\begin{array}{c}
\text{Mod}(\ast) \\
\leftarrow \text{Mod}(\ast) \\
\text{Mod}(\ast)
\end{array}$$

in Mod_2 . By linearity, this is equivalently a morphism in $\text{Mod}(\ast)$ itself. This may hence be regarded as the propagator of the defect TQFT_d^7 .

6 Examples

First, as a simple blueprint for the discussion to follow, in 6.1 we spell out how ordinary linear matrix multiplication arises as a special case of models of the secondary integral transforms in def. 4.18. Then in 6.2 we show as our central example how pull-push (Umkehr maps) in twisted generalized cohomology are another class of examples.

6.1 Ordinary linear algebra – Matrix calculus

Example 6.1 (matrix calculus). Let k be a field, let $\mathbf{H} = \text{Set}$ be the category of sets, and for $X \in \text{Set}$ let $\text{Mod}(X) := k\text{Mod}(X) = \text{Vect}_k(X)$ be the category of X -parameterized vector bundles. This is a model for linear homotopy-type theory by example 3.14. For $X \in \text{FinSet} \hookrightarrow \text{Set}$ a finite set, then an X -dependent linear type $A \in \text{Vect}_k(X)$ is an (unordered) $|X|$ -tuple of vector spaces, where $|X|$ is the cardinality of X . The dependent sum produces the direct sum of these:

$$\sum_X A \simeq \bigoplus_{x \in X} A_x \in \text{Vect}_k.$$

Consider then $X_1, X_2 \in \text{FinSet} \hookrightarrow \text{Set}$ two finite sets of cardinality n_1 and n_2 , respectively, and consider the projection correspondence

$$\begin{array}{ccc}
& X_1 \times X_2 & \\
p_1 \swarrow & & \searrow p_2 \\
X_1 & & X_2
\end{array}$$

Here for $A \in [Y, \text{Vect}_k]$ an n_2 -tuple of vector spaces, then $(p_2)_!(p_2)^*A$ is the n_2 -tuple whose value over $y \in X_2$ is $(A_y)^{\oplus n_1} \simeq A_y \otimes k^{n_1}$. The counit $(p_2)_!(p_2)^*A \rightarrow A$ is the morphism that over each $y \in Y$ is given by forming the sum of n_1 vectors in A_y .

There is an untwisted fiberwise fundamental class on p_2 , given by the canonical choice of identification $k^{n_1} \simeq (k^{n_1})^*$ (“regard row-vectors as column vectors”). With this choice the equivalence of prop. 4.6 is

over $y \in Y$ the induced isomorphism $A_y \otimes k^{n_1} \simeq A_y \otimes (k^{n_1})^*$. The induced fundamental class of def. 4.8 is over each $y \in Y$ the diagonal $A_y \rightarrow (A_y)^{\oplus n_1}$. Dually, the induced measure is over each $y \in Y$ the map $d\mu_{A_y} : \mathbb{D}(A_y^{\oplus n_1}) \rightarrow \mathbb{D}(A_y)$ which is the addition operation on n_1 covectors. This exhibits the canonical “counting measure” on the finite set X_1 .

An $n_1 \times n_2$ -matrix $K \in \text{Mat}_k(n_1, n_2)$ is equivalently a diagram of functors of the form

$$\begin{array}{ccc} & X_1 \times X_2 & \\ p_1 \swarrow & & \searrow p_2 \\ X_1 & & X_2 \\ k_X \searrow & \xleftarrow{K} & \swarrow k_Y \\ & \text{Vect}_k & \end{array} .$$

This defines a (dual) prequantum integral kernel, def. 4.12 between $A_1 = 1_{X_1}$ and $A_2 = 1_{X_2}$ the line bundle on X_1 and X_2 , respectively, with the morphism

$$\xi : (i_2)^* 1_{X_2} = 1_{X_1 \times X_2} \longrightarrow 1_{X_1 \times X_2} = (i_1)^* 1_{X_1}$$

given over $(x, y) \in X_1 \times X_2$ by multiplication with the matrix element $K_{x,y}$.

The induced integral kernel

$$\sum_{X_1} 1_{X_1} \xleftarrow{\sum_{X_1}^{\epsilon}} \sum_{X_1 \times X_2} 1_{X_1 \times X_2} \xleftarrow{\Xi} \sum_{X_1 \times X_2} 1_{X_1 \times X_2} \xleftarrow{\sum_{X_2}^{[i_2]}} \sum_{X_2} 1_{X_2}$$

sends a vector

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n_2} \end{bmatrix} \in \sum_{X_2} 1_{X_2} \simeq k^{n_2}$$

first via the diagonal along X_1 to the image under $\sum_{X_1 \times X_2}$ of

$$\begin{bmatrix} v_1 & v_1 & \cdots & v_1 \\ v_2 & v_2 & \cdots & v_2 \\ \vdots & & & \\ v_{n_2} & v_{n_2} & \cdots & v_{n_2} \end{bmatrix} \in 1_{X_1 \times X_2}$$

then via the integral kernel itself to the image under $\sum_{X_1 \times X_2}$ of

$$\begin{bmatrix} K_{1,1}v_1 & K_{2,1}v_1 & \cdots & K_{n_1,1}v_1 \\ K_{1,2}v_2 & K_{2,2}v_2 & \cdots & K_{n_1,2}v_2 \\ \vdots & & & \\ K_{1,n_2}v_{n_2} & K_{2,n_2}v_{n_2} & \cdots & K_{n_1,n_2}v_{n_2} \end{bmatrix} \in 1_{X_1 \times X_2}$$

and then via summation over X_2 to the image under \sum_{X_1} of

$$\begin{bmatrix} K_{1,1}v_1 + K_{1,2}v_2 + \cdots + K_{1,n_2}v_{n_2} \\ K_{2,1}v_1 + K_{2,2}v_2 + \cdots + K_{2,n_2}v_{n_2} \\ \vdots \\ K_{n_1,1}v_1 + K_{n_1,2}v_2 + \cdots + K_{n_1,n_2}v_{n_2} \end{bmatrix} \in 1_{Y_1} ,$$

hence to the matrix product

$$K \cdot v \in \sum_{X_1} 1_{X_1} \simeq k^{n_1}.$$

6.2 Higher linear algebra – Pull-push in twisted generalized cohomology

The previous example 6.1 considered linear types given by k -modules over sets, for k a commutative ring (a field). This setup has an evident refinement to (stable) homotopy theory, where sets are refined to ∞ -groupoids, commutative rings to E_∞ -rings, and modules to module spectra over these. This homotopy-theoretic refinement of linear algebra used to be advertised as “brave new algebra”, especially when presented in terms of model categories of structured ring spectra. In the intrinsic formulation of ∞ -category theory it is called “higher algebra” in [LurieAlg].

The following example 6.2 shows how twisted Umkehr maps in generalized cohomology as in [ABG10] and section 4.1.4 [Nuiten13] are an example of the general concept of secondary integral transforms in dependent linear homotopy-type theory of def. 4.18.

Example 6.2 (pull-push in twisted generalized cohomology). Let $E \in \mathbf{CRing}_\infty$ be an E_∞ -ring spectrum with ∞ -category of ∞ -modules denoted $E\mathbf{Mod}$. For $X \in \infty\mathbf{Grpd}$ write

$$E\mathbf{Mod}(X) := \mathbf{Func}(X, E\mathbf{Mod})$$

for the ∞ -category of ∞ -functors from X to $E\mathbf{Mod}$. An object in here is sometimes known as an X -parameterized module spectrum, and sometimes as a *local system* of E -modules on X .

For $f : X \rightarrow Y$ a morphism of ∞ -groupoids, there is an induced adjoint triple

$$\left(\sum_f \dashv f^* \dashv \prod_f \right) : E\mathbf{Mod}(X) \begin{array}{c} \xrightarrow{f_!} \\ \xleftarrow{f^*} \\ \xrightarrow{f_*} \end{array} E\mathbf{Mod}(Y),$$

where $f_!$ and f_* are left and right homotopy Kan extension along f , respectively. By example 3.11 this exhibits $E\mathbf{Mod}$ as a linear homotopy-type theory.

For $X, Y, Z \in \infty\mathbf{Grpd}$ three homotopy types, consider a diagram of ∞ -functors of the form

$$\begin{array}{ccc} & Z & \\ g \swarrow & & \searrow f \\ X & & Y \\ \alpha \searrow & \xi & \swarrow \beta \\ & E\mathbf{Mod} & \end{array}.$$

This induces a prequantum integral kernel, def. 4.12, of the form

$$\begin{array}{ccc} & E\mathbf{Mod}(Z) & \\ g_* \swarrow & & \searrow f_* \\ E\mathbf{Mod}(X) & & E\mathbf{Mod}(Y) \\ (p_1)_* \searrow & & \swarrow (p_2)_* \\ & E\mathbf{Mod} & \end{array}$$

with $\xi : f^*\beta \rightarrow g^*\alpha$. Comparison with the discussion in [ABG10] shows that $(p_1)_!\alpha \simeq E_{\bullet+\alpha}(X)$ is the α -twisted E -homology spectrum of X , and $\mathbb{D}((p_1)_!\alpha) \simeq E^{\bullet+\alpha}(X)$ the α -twisted E -cohomology spectrum. Similarly for (Y, β) .

We may decompose the above slice correspondence ξ as

$$\begin{array}{ccccc}
 & & Z & & \\
 & g \swarrow & \downarrow & \searrow f & \\
 X & & & & Y \\
 & \alpha \searrow & \downarrow f^* \beta & \swarrow \beta & \\
 & & E\text{Mod} & &
 \end{array}$$

Consider then the definition for push-forward along the right leg of this diagram the way it appears as def. 4.1.24 in [Nuiten13]. We show that this is a special case of the general def. 4.18.

To that end, notice that in def. 4.1.24 in [Nuiten13] a choice of fundamental class is taken to be a choice of $\gamma \in E\text{Mod}(Y)$ together with an equivalence

$$f_! f^* \beta \xrightarrow{\simeq} \mathbb{D}(f_! f^* \gamma).$$

In the language used here this is a Wirthmüller isomorphism, prop. 4.6, for a choice of fiberwise fundamental class, def. 4.3, under the identification

$$\gamma = \mathbb{D}(\beta \otimes \tau).$$

Indeed, prop. 4.1.27 in [Nuiten13] recovers this identification of τ for the case that f comes from a proper surjective submersion of smooth manifolds; and remark 4.1.28 there observes that when such f is E -orientable then $\gamma = d$ is the degree shift by the dimension of the fibers, as familiar from the classical Poincaré-Thom collapse map.

Then further in def. 4.1.24 in [Nuiten13] the corresponding secondary integral transform is taken to be the composite

$$\begin{array}{ccc}
 E^{\bullet+f^*\beta}(Z) & \xlongequal{\quad} & \mathbb{D}(p_! f_! f^* \beta) \\
 \downarrow \simeq & \searrow d\mu_f & \\
 \mathbb{D}(p_! \mathbb{D}(f_! f^* \gamma)) & \xrightarrow[p_! \mathbb{D}\epsilon_\gamma]{} & \mathbb{D}p_!(\mathbb{D}\gamma) \xlongequal{\quad} E^{\bullet-\gamma}(Y)
 \end{array}$$

Comparison identifies the dashed diagonal composite morphism above indeed as the induced measure $d\mu_f$ in the sense of def. 4.8, as indicated. By the discussion in [Nuiten13] this identifies the secondary integral transform here as given by the twisted Umkehr maps in generalized cohomology due to [ABG11]

$$\int_{i_2} \xi d\mu_{i_2} : E^{\bullet+\alpha}(X) \longrightarrow E^{\bullet+\beta+\tau}(Y),$$

The following example spells out how the construction considered in [HopkinsLurie14] is a special case of the above.

Example 6.3. Let \mathcal{C} be a stable ∞ -category with all limits and colimits, for instance the ∞ -category $E\text{Mod}$ of ∞ -modules over some E_∞ -ring, in which case the following is a special case of example 6.2. For $X, Y \in \infty\text{Grpd}$ two homotopy types and $f : X \rightarrow Y$ a morphism between them, consider the prequantum integral kernel, def. 4.12, given by the correspondence

$$\begin{array}{ccccc}
 & & [X, \mathcal{C}] & & \\
 & f_* \swarrow & & \searrow f_* & \\
 [Y, \mathcal{C}] & & & & [Y, \mathcal{C}] \\
 & \text{id} \searrow & \swarrow = & \swarrow \text{id} & \\
 & & [Y, \mathcal{C}] & &
 \end{array}$$

and by a choice of objects $C, D \in \text{Func}(X, \mathcal{C})$ and a choice of a morphism

$$\xi := f_! u : f_! f^* C \longrightarrow f_! f^* D.$$

Suppose this f is such that it carries a functorial un-twisted fundamental class, hence according to def. 4.8 a natural transformation

$$\mu := [f] : \text{id} \longrightarrow f_! f^*.$$

Then according to def. 4.18 the dual secondary integral transformation induced by this data is the morphism

$$\mathbb{D} \int_f \xi d\mu_f : D \xleftarrow{\epsilon} f_! f^* D \xleftarrow{f_!(u)} f_! f^* C \xleftarrow{\mu} C.$$

This is the notion of integration considered in Notation 4.1.6 of [HopkinsLurie14] (almost exactly denoted there by the same symbols as here, only that we call it the *dual* integration map, following the interpretation in the examples above).

7 Quantization

We now put everything together and discuss how the secondary integral transforms axiomatized in dependent linear homotopy-type theory in 4.3, realized in the model of twisted generalized cohomology theory in 6.2 serve to formalize aspects of quantum field theory.

The way quantum field theory appears in linear homotopy-type theory turns out to be following a principle which in fundamental physics has come to be known as the “holographic principle”, where a quantum field theory of some dimension carries another quantum field theory of one dimension less “on its boundary”. We find this phenomenon appear in two stages:

First, in 7.2, we find non-topological $(d-1)$ -dimensional quantum field theories QFT_{d-1} on the boundary of d -dimensional topological quantum field theories TQFT_d^τ , notably we find ordinary quantum mechanics on the boundary of the non-perturbative 2d Poisson-Chern-Simons theory. This is as in [Nuiten13, Sch13a] and we will be brief. But here the d -dimensional topological quantum field theory is subject to various “twists” and in general there is an obstruction for these twists to consistently orient such as to produce a globally well defined theory. In 7.3 we observe that this consistent global choice of fundamental classes is equivalent to the d -dimensional theory itself being the boundary theory of an (untwisted) $(d+1)$ -dimensional theory TQFT_{d+1} .

Here the obstruction to consistent orientations is also called a *quantum anomaly*, and hence the existence of TQFT_{d+1} is a quantum anomaly cancellation condition.

7.1 TQFT_d^τ via secondary integral transform quantization

A fairly well-known example that has been part of the motivation for a formalization of path integral quantization in terms of pull-push in (generalized) cohomology is the following.

Example 7.1 (string topology operations). Let the ground ring $E = \mathbb{S} \in \text{CRing}_\infty$ be the sphere spectrum and consider the linear homotopy-type theory of example 3.13

$$\begin{array}{c} \text{Sp} \\ \downarrow \\ \text{Grpd}_\infty \end{array},$$

where $\mathbb{S}\text{Mod} \simeq \text{Sp}$ is the ∞ -category of parameterized spectra. Let $X \in \text{Grpd}_\infty$ be the homotopy-type of a compact manifold and consider correspondences induced from 2-dimensional cobordisms

$$\begin{array}{ccc} & \Sigma & \\ \swarrow & & \searrow \\ \Sigma_{\text{in}} & & \Sigma_{\text{out}} \end{array}$$

with non-empty outgoing boundary Σ_{out} as

$$\begin{array}{ccc} & [\Sigma, X] & \\ \swarrow^{(-)|_{\text{in}}} & & \searrow^{(-)|_{\text{out}}} \\ [\Sigma_{\text{in}}, X] & & [\Sigma_{\text{out}}, X] \end{array} .$$

Via example 6.2 the secondary integral transforms of def. 4.12 are given by pull-push of ordinary cohomology in this case. By [CoJo02] these are equivalently the *Chas-Sullivan string topology operations* on the ordinary homology of X . As a system these constitute a 2-dimensional topological quantum field theory TQFT_2 [CoGo03], see also example 4.2.16 and remark 4.2.17 in [Lurie09a].

While example 7.1 stands out as one of the few examples of extended quantum field theories that have already been explicitly constructed by a quantization process from geometric prequantum data, in the traditional formulation this prequantum data is not very explicit.

The following example very briefly recalls how local prequantum field data is encoded in section 3.9.14 of [Sch13a]

Example 7.2 (local Lagrangian prequantum field theory). Let \mathbf{H} be an ∞ -topos equipped with differential cohesion, def. 2.13 whose shape modality \int induces localization at a line object \mathbb{A}^1 . Write

$$U(1) := \mathbb{A}^1 / \mathbb{Z} \in \text{Grp}(\mathbf{H})$$

for the induced circle group. Then for $n \in \mathbb{N}$ there is, by cohesion, canonically a moduli stack $\mathbf{B}^{n+1}U(1)_{\text{conn}} \in \mathbf{H}$ of n -form connections. The slice $\mathbf{H}_{/\mathbf{B}^{n+1}U(1)_{\text{conn}}}$ is the collection of *local Lagrangians* for $(n+1)$ -dimensional local field theory

$$\left[\begin{array}{c} \mathbf{Fields} \\ \downarrow \mathbf{L} \\ \mathbf{B}^{n+1}U(1)_{\text{conn}} \end{array} \right] \in \mathbf{H}_{/\mathbf{B}^{n+1}U(1)_{\text{conn}}} ,$$

where the dependent sum

$$\mathbf{Fields} := \sum_{\mathbf{B}^{n+1}U(1)_{\text{conn}}} \mathbf{L}$$

is interpreted as the moduli stack of fields on which the Lagrangian is defined.

The (concretified) automorphism n -groups of such a local Lagrangian in $\mathbf{H}_{\mathbf{L}}$ encode the Hamilton-de Donder-Weyl equations of motion of the classical local field theory induced by this Lagrangian (this is discussed in section 1.2.11 of [Sch13a], a lightning review is in [Sch13b]).

More generally, given a type of incoming fields $\mathbf{Fields}_{\text{in}} \in \mathbf{H}$ and a type of outgoing fields $\mathbf{Fields}_{\text{out}}$, then a type of field *trajectories* (i.e. paths, histories) from these incoming to these outgoing fields is encoded by a correspondence of the form

$$\begin{array}{ccc} & \mathbf{Fields}_{\text{traj}} & \\ \swarrow^{(-)|_{\text{in}}} & & \searrow^{(-)|_{\text{out}}} \\ \mathbf{Fields}_{\text{in}} & & \mathbf{Fields}_{\text{out}} \end{array} .$$

Now given a ground ring (E_∞ -ring) $E \in \mathbf{CRing}_\infty$, then we say that a choice of *superposition principle* is a choice of representation

$$\rho : \mathbf{B}^n U(1) \longrightarrow \mathrm{GL}_1(E)$$

of the ∞ -group of phases $\mathbf{B}^n U(1)$ on the invertible elements of E . Given this, the local Lagrangian defines a linear homotopy-type depending on **Fields** which is given by the composite

$$\begin{array}{c} \mathbf{Fields} \\ \downarrow \mathbf{L} \\ \mathbf{B}(\mathbf{B}^n U(1)) \\ \downarrow \mathbf{B}\rho \\ \mathbf{BGL}_1(E) \end{array}$$

and which by convenient abuse of notation we denote by the same symbol:

$$\mathbf{L} \in E\mathrm{Mod}(\mathbf{Fields}).$$

In this form one may think of \mathbf{L} equivalently as the local (fully extended) *prequantum line bundle* of the field theory, see [FRS13].

Now with a correspondence given by field trajectories as above, and given local Lagrangians on the incoming and on the outgoing fields, then an extension of this data to a *prequantized Lagrangian correspondence* is a completion to diagram in \mathbf{H} of the form

$$\begin{array}{ccccc} & \mathbf{Fields}_{\mathrm{traj}} & & & \\ & \swarrow \mathrm{in} & & \searrow \mathrm{out} & \\ \mathbf{Fields}_{\mathrm{in}} & & \exp(\frac{i}{\hbar}S) & & \mathbf{Fields}_{\mathrm{out}} \\ & \swarrow \mathbf{L}_{\mathrm{in}} & & \searrow \mathbf{L}_{\mathrm{out}} & \\ & E\mathrm{Mod} & & & \end{array} \quad , \cdot$$

Here the homotopy filling this diagram, which we denote “ $\exp(\frac{i}{\hbar}S)$ ” plays the role of the *action functional* that sends each trajectory to the quantum phase that it induces.

The upshot for the present discussion is that in terms of linear homotopy-type theory such data of a prequantized Lagrangian correspondence is equivalently a prequantum integral kernel in the sense of def. 4.12,

$$\exp(\frac{i}{\hbar}S) : \mathrm{in}^* \mathbf{L}_{\mathrm{in}} \longleftarrow \mathrm{out}^* \mathbf{L}_{\mathrm{out}}$$

on the given trajectory correspondence.

Specific examples of this class of example include example 7.3 below. Further below in 7.3 we see a more fundamental origin of these action functionals.

7.2 QFT_{d-1} via boundary field theory

We now combine all the ingredients and indicate how the cohomological formulation of boundary field theory quantization in [Nuiten13, Sch13a] appears in the formal axiomatization in cohesive linear homotopy-type theory.

- example 7.3 – quantum particle at the boundary of 2d Poisson-Chern-Simons theory;
- example 7.4 – quantum superstring at the boundary of 3d Spin-Chern-Simons theory;

- example 7.5 – D-brane charge and T-duality

The following example indicates how example 7.2 specialises to reproduce traditional non-perturbative geometric quantization (refined from symplectic manifolds to Poisson manifolds).

Example 7.3 (quantum particle at the boundary of 2d Poisson-Chern-Simons theory). Let (X, π) be a Poisson manifold (hence a foliation of symplectic manifolds, hence a foliation of phase spaces of mechanical systems). By the discussion in [FRS13, Bongers13] this canonically induces the local action functional of a 2-dimensional Poisson-Chern-Simons topological field theory

$$\begin{array}{c} \mathrm{SympGrpd}(X, \pi)_{\mathrm{conn}} \\ \downarrow \exp(\frac{i}{\hbar} S_2) \\ \mathbf{B}^2 U(1)_{\mathrm{conn}} \end{array}$$

whose moduli stack of fields is a differential cohomological refinement of what is called the symplectic groupoid $\mathrm{SympGrpd}(X, \pi)$ associated with the Poisson manifold. This is such that perturbatively (on vanishing instanton sectors) it reproduces the Poisson-sigma-model, whose perturbative quantization has been shown by [CaFe00] to yield on its boundary Kontsevich’s perturbative deformation quantization of (X, π) . Here we are after the full, non-perturbative quantization (geometric quantization).

The quantization of $\exp(\frac{i}{\hbar} S_2)$ is supposed to yield a 2-dimensional TQFT $_2^7$. But here we will concentrate only on the boundary conditions of this theory, hence of its higher space of states in codimension 2. For this we may forget the differential refinement and consider just the action on the underlying instanton sectors

$$\begin{array}{c} \mathrm{SympGrpd}(X, \pi) \ . \\ \downarrow \exp(\frac{i}{\hbar} S_2) \\ \mathbf{B}(\mathbf{B}^1 U(1)_{\mathrm{conn}}) \end{array}$$

This restriction classifies what is known as the groupoid pre-quantization of the symplectic groupoid [Bongers13]. From this one observes that the original Poisson manifold is canonical a boundary condition for the 2d Poisson-Chern-Simons theory that it induces, in that there is a morphism in $\mathrm{Corr}(\mathbf{H}_{\mathbf{B}^2 U(1)})$ which in \mathbf{H} is of the form

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ * & & \mathrm{SympGrpd}(X, \pi) \\ \searrow & \swarrow & \swarrow \\ & \mathbf{B}^2 U(1) & \end{array}$$

Now let $E = \mathrm{KU}$ be the complex K-theory E_∞ -ring. By Snaith’s theorem mentioned above this prequantum

correspondence linearizes to a prequantum integral kernel represented by

$$\begin{array}{ccc}
 & X & \\
 \swarrow & & \searrow \\
 * & & \text{SympGrpd}(X, \pi) \\
 \searrow & \swarrow & \swarrow \\
 & \mathbf{B}^2U(1) & \\
 \downarrow & & \\
 & BGL_1(KU) &
 \end{array}$$

via the construction in example 4.14.

Therefore if the right leg can be equipped with a fiberwise fundamental class (typically requiring X to be compact) this correspondence induces a secondary integral transform, def. 4.12, which by example 6.2 is given by pull-push in K-theory as KU-linear map

$$KU \leftarrow KU_{\bullet+\chi}(\text{SympGrpd}(X))$$

which exhibits a class in the complex K-theory of the symplectic groupoid induced by X . In view of remark 4.31 this may be expressed for instance by a composite of bivariant classes in KK-theory. This class may be represented by complex vector spaces associated to the symplectic leaves of X and one finds that these represent the spaces of quantum states under traditional geometric quantization.

More generally, if there is a Lie group G with a Hamiltonian action on X (for instance time evolution for $G = \mathbb{R}$) then the quantization takes place in G -equivariant K-theory and produces a G -equivariant K-theory class of the point, hence a G -representation, being the action of the G -quantum operators on the space of quantum states. For more details see section 5.2 of [Nuiten13].

The following example lifts the previous one up in dimension, from particles to strings.

Example 7.4 (string at the boundary of 3d Chern-Simons theory). The (fully extended) local Lagrangian for 3d Chern-Simons theory is [FSS13]

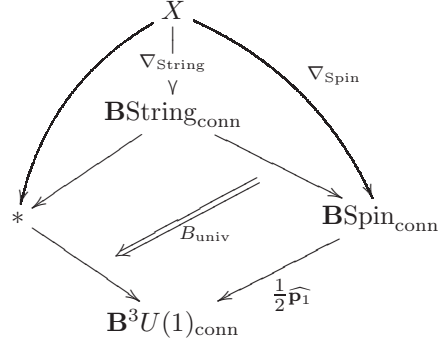
$$\begin{array}{c}
 \mathbf{B}\text{Spin}_{\text{conn}} \\
 \downarrow \frac{1}{2}\widehat{\mathbf{p}}_1 \\
 \mathbf{B}^3U(1)_{\text{conn}}
 \end{array}$$

The *universal* boundary condition for this [Sch13a], namely the homotopy fiber of $\frac{1}{2}\widehat{\mathbf{p}}_1$ is the moduli 2-stack of String-principal 2-connections [FSS12]

$$\begin{array}{ccc}
 & B\text{String}_{\text{conn}} & \\
 \swarrow & & \searrow \\
 * & & B\text{Spin}_{\text{conn}} \\
 \searrow & \swarrow & \swarrow \\
 & B^3U(1)_{\text{conn}} &
 \end{array}$$

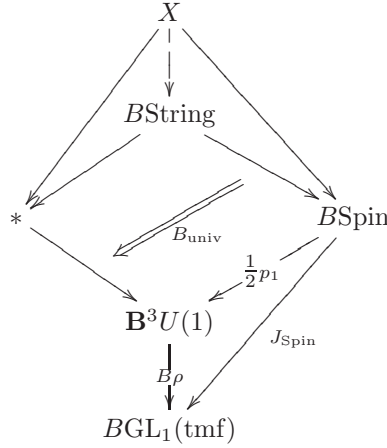
Notice that this means that for a Spin-manifold $\nabla_{\text{Spin}} : X \rightarrow B\text{Spin}$ to constitute a boundary condition is

equivalent to it having (differential) String structure:



Here the total homotopy filling this diagram is physically the twisted Kalb-Ramond B-field on X to which the heterotic string on X couples [FSS09].

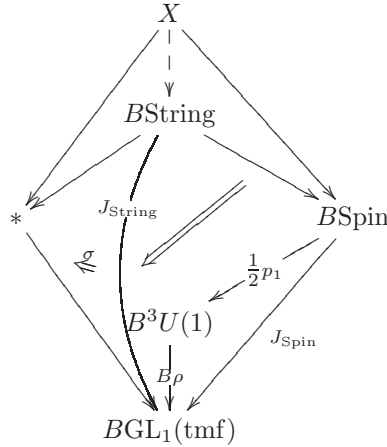
The canonical choice for an E_∞ -ring with a superposition principle in this degree is tmf (see section 8 of [ABG10]).



The corresponding prequantum tmf -line bundle is that classified by the Spin J-homomorphism

$$J_{\text{Spin}} \simeq B\rho \circ \frac{1}{2}p_1$$

(see diagram (8.1) in [ABG10]). Therefore for considering pull-push through this correspondence, it is natural to factor it through the pullback of J_{Spin} to $B\text{String}$, which is the universal String J-homomorphism J_{String} . Such a factorization



is given by the homotopy σ which is the *String orientation of tmf* [AHR10] in its moduli incarnation (this boundary condition correspondence is the lower two third of diagram (8.1) in [ABG10]).

Under quantization the left part of this decomposition becomes, via example 6.2, the tmf-linear map

$$\begin{array}{ccc} & & X^{TX} \wedge \text{tmf} \\ & & \downarrow \\ \text{tmf} = \text{tmf}_\bullet(*) \xleftarrow{\sigma} \text{tmf}_{\bullet+J_{\text{String}}}(B\text{String}) & \simeq & M\text{String} \wedge \text{tmf} \end{array}$$

underlying which is the \mathbb{S} -linear map (we denote all these incarnations by the same symbol “ σ ”, for convenience)

$$\text{tmf} \xleftarrow{\sigma} M\text{String}$$

which is the String orientation as usually given. In [AHR10] it is show that this is on homotopy groups the *Witten genus*. This is the partition function of the heterotic superstring [Wi87] (with target space the given String-spacetime X above).

Example 7.5 (D-brane charge and T-duality). Given a superstring spacetime X equipped with a spin-structure and a Kalb-Ramond B-field $\hat{B} : \rightarrow \mathbf{B}^2U(1)_{\text{conn}}$, then a prequantum D-brane Q in X is a boundary condition for this, hence given by a diagram

$$\begin{array}{ccc} & Q & \\ & \wr & \searrow i \\ * & \xleftarrow{\xi} & X \\ & \searrow \chi & \\ & \mathbf{B}^2U(1) & \end{array}$$

Here the homotopy trivializing this diagram is a trivialization of the class of the background B-field restricted to the brane Q , given by a (twisted) Chan-Paton gauge field ξ on the brane. Via the canonical linearization $\mathbf{B}^2U(1) \rightarrow BGL_1(KU)$ already used in example 7.3 this is naturally quantized over KU -coefficients. For compact Q a fiberwise fundamental class on i exists with twist τ the third integral Stiefel-Whitney class ν_Q of the normal bundle of the inclusion i . This twist vanishes if the normal bundle admits a Spin^c -structure. Given such, the path integral quantization of this boundary is a KU -linear map

$$KU \leftarrow KU_{\bullet+\chi}(X)$$

exhibiting a class in the χ -twisted K-theory of spacetime X . This the *D-brane* of (Q, ξ) in X and the Spin^c -condition is the *Freed-Witten anomaly cancellation* condition for the brane.

More generally a quantum boundary condition according to def. 4.12 is any class ξ in the $(i^*\chi + \nu_Q)$ -twisted K-theory of Q and the *D-brane charge* is the push-forward of that to the χ -twisted K-theory of Q . In this case ξ is a non-abelian Chan-Paton gauge bundle on Q and the fact that it is in $(i^*\chi + \nu_Q)$ -twisted K-theory is the *Freed-Witten-Kapustin anomaly cancellation*. For more see section 5.2 4 of [Nuiten13].

Now suppose that X is a torus-fiber bundle over some base Y . Then a (topological) T-duality transfor-

mation is a correspondence of the form

$$\begin{array}{ccccc}
& & X \times_Y \tilde{X} & & \\
& \swarrow & & \searrow & \\
X & & & & \tilde{X} \\
& \swarrow & \text{---} & \searrow & \\
& & Y & & \\
& \swarrow & & \searrow & \\
& & \mathbf{B}^2 U(1) & & \\
& & \downarrow & & \\
& & BGL_1(KU) & &
\end{array}$$

χ $\tilde{\chi}$

where $\tilde{X} \rightarrow Y$ is another torus bundle spacetime with another B-field $\tilde{\chi}$, and the homotopy filling this diagram is a (twisted) line bundle on the torus fiber product $X \times_Y \tilde{X}$ with the property that on each fiber it restricts to the canonical Poincaré-line bundle on the product of a torus and its dual, up to equivalence. The fundamental theory of topological T-duality says, in our language here, that the path integral quantization of this correspondence induces an equivalence

$$KU_{\bullet+\chi} \xleftarrow{\simeq} KU_{\bullet+\tilde{\chi}}$$

between the twisted K-theories of the two T-dual string backgrounds, hence between their D-brane charges. In terms of KK-theory this is discussed in [BMRS07].

According to example 7.4 the above correspondences should be regarded as correspondences-of-correspondences between those in example 7.4, where the B-field is not a map to $BGL_1(KU)$ but a homotopy between maps to $BGL_1(\mathrm{tmf})$. This would mean that T-duality then induces an equivalence of non-perturbative Witten genera, namely of tmf -classes of the String-spacetimes X and \tilde{X} . This has indeed been found to be the case by Thomas Nikolaus for a fairly large class of cases¹⁷.

7.3 TQFT_{d+1} via quantum anomaly cancellation

Given a correspondence

$$X_1 \xleftarrow{i_1} Z \xrightarrow{i_2} X_2$$

we defined in def. 4.18 an integral kernel based on this correspondence to be data of the form

$$\xi : i_1^* A_1 \longleftarrow i_2^* A_2 .$$

One may ask where this form of data comes from. In example 4.14 and then more specifically in example 7.2 we gave a class of constructions that occur naturally in practice which do yield this kind of data. But here we want to go one step further and understand this data as being in turn the boundary field theory data of a TQFT of yet one more dimension higher.

Moreover, so far the TQFT_d^τ which we obtained correspondence-wise by quantization via secondary integral transforms may be “anomalous” in that its correspondence-wise construction does not actually extend to a monoidal functor

$$\mathrm{TQFT}_d^\tau : \mathrm{Bord}_n \xrightarrow{\exp(\frac{i}{\hbar} S) d\mu} \mathrm{Corr}(\mathbf{H}) \xrightarrow{f(-) d\mu} \mathrm{Mod}(\ast) .$$

¹⁷See Nikolaus’ talk at the String Geometry Network meeting.

Here we show that the condition that TQFT_d^τ is quantum anomaly free means that it is itself the boundary field theory of yet another TQFT_{d+1} .¹⁸

Definition 7.6. For $\text{Mod}(-) \rightarrow \mathbf{H}$ a model for linear homotopy-type theory, which satisfies Beck-Chevalley, def. 5.5, and is 2-monoidal, def. 5.7, write

$$\text{TQFT}_{d+1} : \text{Corr}_1(\mathbf{H}) \longrightarrow \text{Mod}_2$$

for the $(\infty, 1)$ -functor from the $(\infty, 1)$ -category of correspondences in \mathbf{H} to the $(\infty, 2)$ -category of 2-modules, def.5.4, given by sending homotopy-types X to their ∞ -categories $\text{Mod}(X)$ of linear homotopy-types dependent on them, and sending correspondences $X_1 \xleftarrow{i_1} Z \xrightarrow{i_2} X_2$ as above to their *linear polynomial functors*

$$\text{Mod}(X_1) \xleftarrow{\sum_{i_1} \circ i_2^*} \text{Mod}(X_2)$$

as in def. 3.20.

Remark 7.7. That def. 7.6 indeed gives a monoidal $(\infty, 2)$ -functor is the content of cor. 5.10.

Proposition 7.8. *In a linear homotopy-type theory which satisfies Beck-Chevalley, def. 5.5, and is 2-monoidal, def. 5.7, then the functor TQFT_{d+1} in def. 7.6 is monoidal.*

Proof. By assumption of 2-monoidality it suffices to see that for $X_1 \xleftarrow{i_1} Z \xrightarrow{i_2} X_2$ and $\tilde{X}_1 \xleftarrow{\tilde{i}_1} \tilde{Z} \xrightarrow{\tilde{i}_2} \tilde{X}_2$ two correspondences in \mathbf{H} , and $(p_1^* A) \otimes (p_2^* \tilde{A}) \in \text{Mod}(X_2 \times \tilde{X}_2)$, then

$$\sum_{i_1 \times \tilde{i}_1} (i_2 \times \tilde{i}_2)^* \left((p_1^* A) \otimes (p_2^* \tilde{A}) \right) \simeq \left(p_1^* \sum_{i_1} i_2^* A_1 \right) \otimes \left(p_2^* \sum_{\tilde{i}_1} \tilde{i}_2^* \tilde{A} \right).$$

Given the assumption of Beck-Chevalley, this is the statement of prop.5.9. \square

We now consider boundary conditions for TQFT_{d+1} . For that purpose write

$$1_{d+1} : \text{Corr}(\mathbf{H}) \longrightarrow \text{Mod}_2,$$

for the $(\infty, 2)$ -functor which sends every correspondence to the identity functor on $\text{Mod}(*)$.

Proposition 7.9. *A $\text{Mod}(*)$ -linear natural transformation*

$$\exp\left(\frac{i}{\hbar} S\right) : 1_{d+1} \longrightarrow \text{TQFT}_{d+1}$$

is over each correspondence $X_1 \leftarrow Z \rightarrow X_2$ equivalently a prequantum integral kernel, def. 4.12.

Proof. Consider the naturality square

$$\begin{array}{ccc} \text{Mod}(*) & \xleftarrow{\text{id}} & \text{Mod}(*) \\ \downarrow 1_* \mapsto A_1 & \swarrow \exp\left(\frac{i}{\hbar} S\right) & \downarrow 1_* \mapsto A_2 \\ \text{Mod}(X_1) & \xleftarrow{\sum_{i_1} \circ i_2^*} & \text{Mod}(X_2) \end{array}.$$

Here by $\text{Mod}(*)$ -linearity the vertical functors are fixed by their image of the tensor unit, which we denote by A_1, A_2 respectively. Therefore the unit component of this natural transformation on this A_2 has to be a morphism in $\text{Mod}(X_1)$ of the form

$$\exp\left(\frac{i}{\hbar} S\right)_{A_2} : \sum_{i_1} i_2^* A_2 \longrightarrow A_1.$$

¹⁸The result here is joint with Joost Nuiten.

On a general object $\tau \in \text{Mod}(\ast)$ the component of the transformation has to be of the form

$$\exp(\frac{i}{\hbar}S)_{A_2 \otimes \tau} : \sum_{i_1} (i_2^* A_2 \otimes Z^* \tau) \simeq \sum_{i_1} (i_2^* A_2 \otimes i_1^* X_1^* \tau) \longrightarrow A_1 \otimes X_1^* \tau$$

which by Frobenius reciprocity, def. 3.2, is equivalently of the form

$$\exp(\frac{i}{\hbar}S)_{A_2 \otimes \tau} : (\sum_{i_1} i_2^* A_2) \otimes X_1^* \tau \longrightarrow A_1 \otimes X_1^* \tau.$$

By linearity this is fixed to be $\exp(\frac{i}{\hbar}S)_{A_2 \otimes \tau} \simeq \exp(\frac{i}{\hbar}S)_{A_2} \otimes \text{id}_\tau$ and hence the transformation is equivalent to the data consisting of A_1 , A_2 and $\exp(\frac{i}{\hbar}S)_{A_2}$. Finally observe that by the $(\sum_{i_1} \dashv i_1^*)$ -adjunction the datum $\exp(\frac{i}{\hbar}S)_{A_2}$ is equivalently given by its adjunct $\xi : i_2^* A_2 \longrightarrow i_1^* A_1$, which is the integral kernel in question. \square

Definition 7.10. Given a choice of untwisted fiberwise fundamental class on i_2 , def. 4.3, consider the transformation

$$\mathbb{D} \int (-) d\mu : \text{TQFT}_{d+1} \longrightarrow 1_{d+1}$$

restricted to the given correspondence $X_1 \xleftarrow{i_1} Z \xrightarrow{i_2} X_2$ whose component there is

$$\begin{array}{ccc} \text{Mod}(X_1) & \xleftarrow{\sum_{i_1} \circ i_2^*} & \text{Mod}(X_2) \\ \sum_{X_1} \downarrow & \swarrow \sum_{X_2} [i_2] & \downarrow \sum_{X_2} \\ \text{Mod}(\ast) & \xleftarrow{\text{id}} & \text{Mod}(\ast) \end{array},$$

where the transformation filling this diagram is the X_2 -dependent sum of the given fundamental class $[i_2]$ on i_2 , def. 4.8.

Combining this we obtain a twisted

$$\text{TQFT}_d^\tau := \mathbb{D} \int \exp(\frac{i}{\hbar}S) d\mu : \text{Corr}(\mathbf{H}) \longrightarrow \text{Mod}(\ast)$$

as the unit component of the composite of these two transformations, hence as a defect from the trivial $(d+1)$ -dimensional theory to itself. It sends a correspondence to the unit component of the pasting composite of natural transformations as follows

$$\begin{array}{ccc} \text{Mod}(\ast) & \xleftarrow{\text{id}} & \text{Mod}(\ast) \\ 1_\ast \mapsto A_1 \downarrow & \swarrow \exp(\frac{i}{\hbar}S) & \downarrow 1_\ast \mapsto A_2 \\ \text{Mod}(X_1) & \xleftarrow{\sum_{i_1} \circ i_2^*} & \text{Mod}(X_2) \\ \sum_{X_1} \downarrow & \swarrow \sum_{X_2} [i_2] & \downarrow \sum_{X_2} \\ \text{Mod}(\ast) & \xleftarrow{\text{id}} & \text{Mod}(\ast) \end{array}.$$

Proposition 7.11. *The unit component of this pasting composite*

$$\mathbb{D} \int \exp(\frac{i}{\hbar}S) d\mu : 1_{d+1} \xrightarrow{\exp(\frac{i}{\hbar}S)} \text{FQFT}_{d+1} \xrightarrow{\mathbb{D} \int (-) d\mu_{i_2}} 1_{d+1}$$

is the dual secondary integral transform

$$\mathbb{D} \int_Z \xi d\mu_{i_2} : \sum_{X_1} A_1 \longleftarrow \sum_{X_2} A_2$$

which is associated by def. 4.18 to the integral kernel ξ corresponding to $\exp(\frac{i}{\hbar}S)$ via the proof of prop. 7.9.

Proof. The pasting natural transformation here has as unit component the map

$$\sum_{X_1} A_1 \xleftarrow{\sum_{X_1} \exp(\frac{i}{\hbar}S)} \sum_{X_1} \sum_{i_1} i_2^* A_2 \xleftarrow{\sum_{X_2} [i_2]} \sum_{X_2} A_2$$

By the general formula for adjoints we have that ξ and $\exp(\frac{i}{\hbar}S)$ are related by

$$\exp(\frac{i}{\hbar}S) : A_1 \xleftarrow{\epsilon} \sum_{i_1} i_1^* A_1 \xleftarrow{\sum_{i_1} \xi} \sum_{i_1} i_2^* A_2 .$$

Inserting this into the first expression manifestly yields the secondary integral transform formula of def. 4.18, up to canonical equivalence. \square

Remark 7.12. (consistent orientations and quantum anomalies) Proposition 7.11 provides a succinct formulation of what it takes to choose fiberwise fundamental classes, def. 4.3, on a system of correspondences *consistently*, namely such that the operation of secondary integral transforms is functorial in the correspondences: the condition is that $\int(-)d\mu : \text{FQFT}_{d+t} \longrightarrow 1_{d+1}$ is indeed a natural transformation, hence indeed a boundary condition for the tautological $(d+1)$ -dimensional theory. The existence of such consistent orientations is the central obstruction to the existence of the quantization process, and such obstructions to quantization are known in the physics literature as *quantum anomalies*. A clean account of quantum anomalies as traditionally considered is in [Fre86]; for quantum anomalies from the perspective as considered here see also [Fre12]. Therefore finding consistent orientations is *quantum anomaly cancellation*. The problem of finding consistent orientations for integral transforms given by pull-push had previously been highlighted in [FHT07] for the special case of pull-push in equivariant K-theory.

8 Conclusion and Outlook

8.1 Holographic principle

Ever since the widely (and wildly) cited articles [Maldacena97, Wi98] it has become common in fundamental physics to conjecture that topological/gravitational quantum field theories TQFT_{d+1}^τ of dimension d carry on their boundary, as *boundary field theories* non-topological quantum field theories QFT_{d-1} , such that the boundary fields of TQFT_d^τ encode the correlators of QFT_{d-1} . This relation has come to be known as the *holographic principle*, and pairs of field theories related this way as *holographic duals*. With all the ambition that goes into these conjectures, it is often underappreciated that there has previously been a seminal example of field theories for which such a holographic relation is an established theorem. This is the relation between 3-dimensional Chern-Simons theory for a compact gauge group G , and the 2-dimensional Wess-Zumino-Witten model, describing the propagation of a quantum string on G (see [Ga99] for a survey of both systems, and specifically see around p. 30 for their “holographic” relation).

Often such holographic relations sit in longer hierarchies of boundary field theories see notably [Sa11] and [Wi11]. In particular the d -dimensional topological field theory TQFT_d^τ may be anomalous, with its

quantum anomaly encoded by it being the (partial) boundary of some TQFT_{d+1} . At the pre-quantum level a formalization of such hierarchies of field theories has been discussed in section 3.9.14 of [Sch13a].

Comparison shows that the quantization process via secondary integral transforms discussed here reflects this holographic form. By prop. 7.11 and via the examples in 7.2 it produces $(d-1)$ -dimensional quantum field theories QFT_{d-1} as boundary conditions of d -dimensional topological quantum field theories TQFT_d^τ , which however in general involve an intricate twisting, hence a “quantum anomaly”. The consistent orientation of the twists and hence the “cancelling of the quantum anomaly” is equivalent to TQFT_d^τ in turn being a (two-sided) boundary for a $(d+1)$ -dimensional (untwisted) TQFT_{d+1} .

field theory	spaces of states	propagator
TQFT_{d+1}	$\text{Mod}_2 \in \text{Cat}_2$	integral transform
TQFT_d^τ	$\text{Mod}(\ast) \in \text{Mod}_2$	secondary integral transform, path integral
QFT_{d-1}	$\sum_X A_X \in \text{Mod}(\ast)$	equivariance under Hamiltonian group action

We might informally depict this situation as follows

$$\text{QFT}_{d-1} \xrightarrow{\partial} \text{TQFT}_d^\tau \xrightarrow{\partial} \text{TQFT}_{d+1}$$

One famous example of such a hierarchy of “holographic” relations is supposed to be the between 3-dimensional Chern-Simons theory CS_3 and the 2-dimensional Wess-Zumino-Witten model WZW_2

$$\text{WZW}_2 \xrightarrow{\partial} \text{CS}_3 \xrightarrow{\partial} \text{TQFT}_4 .$$

Another expected example of much current interest is the relation between the 7-dimensional Chern-Simons theory CS_7 that appears as the compactification of the 11-dimensional Chern-Simons term in 11-dimensional supergravity [FSS12] on whose boundary the $\text{AdS}_7/\text{CFT}_6$ correspondence (see [Na07]) predicts the 6-dimensional $(2,0)$ -supersymmetric field theory WZW_6

$$\text{WZW}_6 \xrightarrow{\partial} \text{CS}_7 \xrightarrow{\partial} \text{TQFT}_8 .$$

These two examples have also been amplified in [Fre12].

Notice that WZW_6 is thought to become 4-dimensional(super-)Yang-Mills theory after Kaluza-Klein reduction on a torus whose modulus is the combined Yang-Mills coupling constant and θ -angle [Wi07b]. (Under this reduction Möbius transformations of the torus are supposed to yield the S-duality of super-Yang-Mills.) Therefore there is a plausible route that connects the quantization process described here to one of the core problems of quantization theory, that of Yang-Mills theory [JaWi].

8.2 Motives

We conclude here with some comments revolving around the observation that the quantization process found in linear homotopy-type theory above is broadly analogous to the construction of categories of *motives* (see for instance [Mazur04] for the general idea of motives and see [Le06] for a technical introduction).

Consider specifically the model of linear homotopy-type theory in example 3.11 above, given by bundles of E -module spectra over ∞ -groupoids (“local systems” with coefficients in E -modules), for $E \in \text{CRing}_\infty$ some E_∞ -ring. As we have seen in example 6.2 in this case a prequantum integral kernel, def. 4.12, consists of

1. a correspondence of homotopy types of spaces

$$\begin{array}{ccc}
 & Z & \\
 & \downarrow (i_1, i_2) & \\
 X_1 & \times X_2 & \\
 \swarrow p_1 & & \searrow p_2 \\
 X_1 & & X_2
 \end{array} ;$$

2. a choice of twists χ_i of E -cohomology on X_i ;
3. a fiberwise E -linear map

$$\xi : i_1^* \chi_1 \longrightarrow i_2^* \chi_2$$

on the correspondence space Z inducing an E -linear map

$$\Xi := \int_Z \xi d\mu : E_{\bullet+\chi_1}(X_1) \longrightarrow E_{\bullet+\chi_2}(X_2)$$

between the twisted homology spectra.

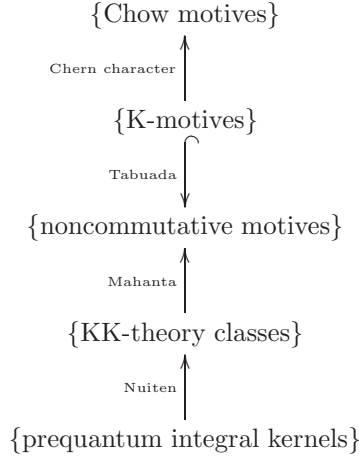
First notice that here in the special case that $E_{\bullet+\chi_1}(X_1)$ happens to be equivalent to $E_{\bullet}(\ast)$ (for instance because $X_1 \simeq \ast$), then $\Xi \in E_{\bullet+\chi_2}(X_2)$ is equivalently just a cycle in the χ_2 -twisted E -homology of X_2 . Conversely if the left hand spectrum is equivalent to $E_{\bullet}(\ast)$ then $\Xi \in E^{\bullet+\chi_1}(X_1)$ is equivalently a cocycle in the χ_1 -twisted E -cohomology of X_1 . In the general case we may speak of Ξ here as being a (χ_1, χ_2) -twisted cocycle in *bivariant* E -cohomology. Notice furthermore that in the case that the twists vanish then the E -cohomology of a homotopy type carries the structure of an E_{∞} -ring and so one may think in this case of Ξ as exhibiting the cohomology of the correspondence space as a bimodule over the cohomology of its legs.

This form of a datum given by a correspondence of spaces equipped with cycles on the correspondence space is familiar from the definition of (pure or mixed) *motives*, where the spaces considered are (projective) smooth schemes and the cocycles are algebraic cycles (see section 8 of chapter 1 of [CCM05] for a review). The literature also considers correspondences of schemes equipped instead with cycles in algebraic K-theory (see section 5.3 of [Ta11a] for a review). These map to pure (Chow-)motives via the algebraic Chern character and faithfully embed into *non-commutative motives* ([Ta11a], p. 9,10) where the role of the cycles is played by bimodules over algebraic data associated with the left and the right leg of the correspondence (see [Ta11b] for a survey).

Indeed, in the special case of linear homotopy-type theory over $E = \text{KU}$ the complex K-theory spectrum (as in the examples in 7.2 above) and restricting attention to homotopy types X_i and Z of suitable compact manifolds, then the bivariant cocycles Ξ above may be equivalently expressed as classes in bivariant operator K-theory, i.e. KK-theory (section 4.2 of [Nuiten13]). It had long been suggested by Connes (originally in section 3 of [CoSk84], for more see section 1 of chapter 4 of [CCM05]) that KK-theory (or its pseudo-abelian envelope given by cyclic cohomology) is to be regarded as the analog in non-commutative topology (C^* -algebra theory) of the theory of motives in algebraic geometry. Recently in [Mahanta13] this analogy has been substantiated by the construction of a canonical map from KK-theory to non-commutative motives.

In summary this means that there is a zig-zag of comparison functors relating motives in algebraic

geometry to prequantized integral kernels in linear homotopy-type theory as considered here:



So far this is just the pre-quantum level. When we now consider quantization, then the relation to motivic theory goes further still. Above in 7.2 we discussed the structure of a “consistent orientation” or “anomaly free densitized action $\exp(\frac{i}{\hbar}S)d\mu$ ” as the necessary structure to make the secondary integral transform operation of “pull-tensor-push” through such motivic correspondences, def. 4.18 well defined, such as to yield a functor

$$\text{QFT}_d = \int \exp(\frac{i}{\hbar}S)d\mu : \text{Corr}(\mathbf{H}) \longrightarrow E\text{Mod}(\ast)$$

from correspondences to E -modules over the point. In example 4.19 we saw that given such a functor, it is of interest to consider its restriction along the inclusion

$$\mathbf{H} \longrightarrow \text{Corr}(\mathbf{H})$$

which sends morphisms in \mathbf{H} to “right way”-morphisms. Conversely therefore one may consider equipping a given functor $\mathbf{H} \longrightarrow E\text{Mod}$ with the structure of a lift to a functor on correspondences. (Notice that we are considering homology-valued QFTs, but by postcomposing with E -duality they become cohomologically valued functors on \mathbf{H}^{op}). Such extensions of (contravariant) functors on the given category of geometric spaces to functors on correspondences equipped with cycles are known as *sheaves with transfer* in the theory of motives. (Notice that, by remark 3.5, the concept of “transfer” here is indeed at least closely related to the concept of Umkehr maps via fundamental classes that we consider.) Those complexes of sheaves with transfer on smooth schemes that are only sensitive to the \mathbb{A}^1 -homotopy type of schemes form the category of “effective geometric motives” (see section 3.1 of [Vo00] or section 3 of lecture 2 of [Le06]).

Therefore, translated this way to the present setup (and imposing that the presheaves are monoidal functors) effective geometric motives are analogous to QFTs obtained by quantization via secondary integral transforms as discussed here.

While we feel that the canonical constructions in linear homotopy-type theory that we considered here exhibits this analogy between quantization and motives in a pleasingly holistic way, at least various aspects of such a relation between motivic structures and quantization have a long tradition in the literature, often implicitly. In the context of mechanics it goes back to [We71] and in the context of field theory it was amplified in [Fre92]. A fairly comprehensive and commented list of references does not fit into the present note, but may be found online at ncatlab.org/nlab/show/motivic+quantization#PrevLit.

Notice that in the approximation to quantization given by perturbation theory (formal algebraic deformation quantization) a deep relation to motivic structures was suggested in [Ko99], where the space of choices of formal deformation quantizations is argued to be a torsor over a quotient of the motivic Galois

group equivalent to the Grothendieck-Teichmüller group. A precise version of this statement was proven in [Do11].

Since our discussion here concerns the full (non-perturbative, geometric) quantization (as in example 7.3 above) it seems natural to expect that the motivic structures that appear here may be related to these seen in formal deformation quantization. This remains to be explored.

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